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A Minimization Method for AND-EXOR Expressions Using Lower Bound Theorem

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SUMMARY

This paper presents properties of Exclusive-OR Sum-of-Products expressions (ESOPs) and their minimization algorithm. First, lower bounds on the number of products in minimum ESOPs (MESOPs) are shown. Then an algorithm to simplify ESOPs is presented. In most cases, the algorithm proves their minimality for functions of up to five variables. It utilizes a table of MESOPs for all the functions of four variables, and: 1) finds a lower bound on the number of products in the MESOP; 2) obtains an initial solution for the given function; 3) simplifies the ESOP by an iterative improvement method; and 4) stops the iterative improvement when the solution is proved to be minimum. Experimental results show that this algorithm proves the minimality of about 98 percent of the five-variable functions.

Key words: Combinational circuit, Exclusive-OR sum-of-products, logic minimization, complexity.

1. Introduction

Recently, logic synthesis tools have been used to design LSI circuits. Such tools often produce better solutions in a shorter time than manual design, and are now indispensible in the practical LSI design. Most logic synthesis tools utilize sum-of-products expression (SOP) minimizers extensively in the programs. However, arithmetic circuits can be realized with many fewer gates if EXOR gates as well as AND and OR gates are available. Thus, a logic synthesis tool for the circuits utilizing EXOR gates will be very useful.

To design multilevel circuits with EXOR gates, an efficient minimizer for exclusive-OR sum-of-products expression (ESOP) is necessary. As for SOPs, efficient minimization algorithms exist. As for exact minimization of ESOPs, only exhaustive [1, 10] or virtually exhaustive methods [15] are known. As for near minimum ESOPs, several heuristic algorithms have been developed [2-4].

In this paper, we present methods to find lower bounds on the number of products in MESOPs for the functions of n variables by using the MESOPs for the functions of $(n - 1)$ variables. We show also a simplification algorithm for ESOPs with five-variables. This algorithm simplifies ESOPs and proves their minimality for about 98 percent of the five-variable functions. For the functions with five or more variables, the exhaustive method would be impractical because the number of the combinations to consider is too large. Thus, this is the first algorithm to guarantee the

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minimality of the solution, although it is restricted to the five-variable functions. Also, when this algorithm is extended for the functions with six or more variables, we can obtain ESOPs with fewer products, and simplify ESOPs quickly.

2. Definitions and Basic Properties of Minimum ESOPs

Definition 1. x and $\bar{x}$ are *literals* of a variable x.

Definition 2. Let $S_i \subseteq \{0, 1\}$ and $S_i \neq 0$ ($i = 1, 2, \cdots, n$). $T = x_1^{S1} x_2^{S2} \cdots x_n^{Sn}$ is a *product term*, where $x_i^{\{0\}} = \bar{x}_i$, $x_i^{\{1\}} = x_i$, $x_i^{\{0,1\}} = 1$ and $x_i^{0} = 0$. For simplicity, $x_i^{\{0\}}$, $x_i^{\{1\}}$ and $x_i^{\{0,1\}}$ are denoted by x_i^0, x_i^1 and x_i^2, respectively.

Definition 3. A sum-of-product form

$$\sum_{(S_1,S_2,\ldots,S_n)} \oplus x_1^{S_1} x_2^{S_2} \ldots x_n^{S_n}$$

is an *Exclusive-or Sum-of-Products expression (ESOP)*.

Remark 1. Up to now, various classes of AND-EXOR expressions, such as positive polarity Reed-Muller expressions, fixed polarity Reed-Muller expressions, Kronecker expressions, pseudo-Kronecker expressions and generalized Reed-Muller expressions, have been proposed [5-7, 17]. Among these expressions, ESOPs are the most generalized ones. Therefore, the number of the products in ESOPs can be minimum.

Definition 4. An ESOP for f is said to be a *minimum ESOP* (or *MESOP*) if the number of products is the minimum.

Definition 5. The number of products in an ESOP F is denoted by $\tau(f)$. The number of products in a MESOP for f is denoted by $\tau(f)$.

Theorem 1. Let a function be represented as $f = x^* \cdot g$, where g is a function not dependent on the variable x and x^* represents $\bar{x}$ or x. Then, $\tau(f) = \tau(g)$.

Proof.

(1) Let G be a MESOP for g. Note that $x^* \cdot G$ represents f. Also, note that $\tau(x^* \cdot G) = \tau(g)$. Therefore, we have $\tau(f) \leq \tau(g)$.

(2) Let a MESOP for f be F. In F, if we set x to 1 when $x^* = x$ and x to 0 when $x^* = \bar{x}$, then F represents the function g. Therefore, we have $\tau(g) \leq \tau(f)$. From items (1) and (2), we have the theorem.

Q.E.D.

Theorem 2. If a function f can be represented as $f = g \oplus h$, then $\tau(f) \leq \tau(g) + \tau(h)$.

Proof. Let G and H be MESOPs for functions g and h, respectively. Because $G \oplus H$ represents the function f, we need at most $\tau(G) + \tau(H)$ products to realize f. Q.E.D.

3. Lower Bounds on the Number of Products in MESOPs

In this section, some lower bounds on the number of products in MESOPs for a given function are considered. These results are useful for the simplification of ESOPs.

Lemma 1. $\tau(f) \geq L1$, where $L1 = \tau(0:1)$, $\tau(0:1) = \tau(f(0) \oplus f(1))$, $f(a) = f(a, x_2, x_3, \cdots, x_n)$ and $a \in \{0, 1\}$.

(Proof is in Appendix 1.)

Lemma 2. $\tau(f) \geq L2$, where $L2 = \{\tau(0, 0:0, 1)\} + \tau(0, 0:1, 0) + \tau(1, 1:0, 1) + \tau(1, 1:1, 0)\}/2$, $\tau(a, b:c, d) = \tau(f(a, b) \oplus f(c, d))$, $f(a, b) = f(a, b, x_3, x_4, \cdots, x_n)$ and $a, b, c, d \in \{0, 1\}$.

(Proof is in Appendix 2.)

Lemma 3. $\tau(f) \geq L3$, where $L3 = \{\tau(0, 0, 0:0, 0, 1) + \tau(0, 0, 0:0, 1, 0) + \tau(0, 0, 0:1, 0, 0) + \tau(0, 1, 1:0, 0, 1) + \tau(0, 1, 1:0, 1, 0) + \tau(0, 1, 1:1, 1, 1) + \tau(1, 0, 1:0, 0, 1) + \tau(1, 0, 1:1, 0, 0) + \tau(1, 0, 1:1, 1, 1) + \tau(1, 1, 0:0, 1, 0) + \tau(1, 1, 0:1, 0, 0) + \tau(1, 1, 0:1, 1, 1)\}/4$, $\tau(a, b, c:d, e, h) = \tau(f(a, b, c) \oplus f(d, e, h))$, $f(a, b, c) = f(a, b, c, x_4, x_5, \cdots, x_n)$ and $a, b, c, d, e, h \in \{0, 1\}$.

(Proof is in Appendix 3.)

Theorem 3. $\tau(f) \geq A$, where $A = \max_i[\max\{\tau(f_{i0}), \tau(f_{i1}), \tau(f_{i2})\}]$, $f = \bar{x}_i \cdot f_{i0} \oplus x_i \cdot f_{i0} = f(x_i = 0)$, $f_{i1} = f(x_i = 1)$, $f_{i2} = f_{i0} \oplus f_{i1}$ and $i = 1, 2, \cdots, n$.

Proof. Let a MESOP for f be

$$F_m = \bar{x}_i \cdot F_a \oplus x_i \cdot F_b \oplus F_c \quad (1)$$

From Eq. (1), we have

$$\tau(f) = \tau(F_a) + \tau(F_b) + \tau(F_c) \quad (2)$$

By setting x_i to 0 in Eq. (1), we have

$$F_m(x_i = 0) = F_a \oplus F_c \quad (3)$$

Because Eq. (3) represents the function f_{i0}, we have

$$\tau(f_{i0}) \leq \tau(F_a) + \tau(F_c) \leq \tau(f) \quad (4)$$

Similarly, by setting x_i to 1 in Eq. (1), we have

$$F_m(x_i = 1) = F_b \oplus F_c \quad (5)$$

Because Eq. (5) represents the function f_{i1}, we have

$$\tau(f_{i1}) \leq \tau(F_b) + \tau(F_c) \leq \tau(f) \quad (6)$$

By (3) $\oplus$ (5), we have $f_{i0} \oplus f_{i1} = F_a \oplus F_b$. Therefore, we have

$$\tau(f_{i2}) \leq \tau(F_a) + \tau(F_b) \leq \tau(f) \quad (7)$$

From Eqs. (4), (6) and (7), we have

$$\max\{\tau(f_{i0}), \tau(f_{i1}), \tau(f_{i2})\} \leq \tau(f) \quad (8)$$

This relations holds for all possible i, thus we have the theorem. Q.E.D.

Theorem 4. $\tau(f) \leq B$, where $B = \min_i[B_i]$, $B_i = \tau(f_{i0}) + \tau(f_{i1}) + \tau(f_{i2}) - \max\{\tau(f_{i0}), \tau(f_{i1}), \tau(f_{i2})\}$, $f_{i0} = f(x_i = 0)$, $f_{i1} = f(x_i = 1)$, $f_{i2} = f_{i0} \oplus f_{i1}$ and $i = 1, 2, \cdots, n$.

Proof. Because f can be represented as $f = f_{i0} \oplus x_i \cdot f_{i2} = f_{i1} \oplus \bar{x}_i \cdot f_{i2} = \bar{x}_i \cdot f_{i0} \oplus x_i f_{i1}$, we have $\tau(f) \leq \tau(f_{i0}) + \tau(f_{i2})$, $\tau(f) \leq \tau(f_{i1}) + \tau(f_{i2})$ and $\tau(f) \leq \tau(f_{i0}) + \tau(f_{i1})$. Therefore, we have $\tau(f) \leq B_i$. Because these relations hold for all possible i, we have the theorem. Q.E.D.

Lemma 4. Suppose that f is represented as $f = x_i \cdot f_{i0} \oplus x_i \cdot f_{i1}$, where $f_{i0} = f(x_i = 0)$, $f_{i1} = f(x_i = 1)$. If $\tau(f) = \tau(f_{i1})$, then a MESOP for f has a form $x_i \cdot F_d \oplus F_e$, where F_d and F_e are ESOPs not containing the variable x_i.

Proof. Suppose that a MESOP for f is represented in the form:

$$x_i \cdot F_d \oplus F_e \oplus \bar{x}_i \cdot F_g \quad (9)$$

where $F_g \neq 0$. Then, we have

$$\tau(f) = \tau(F_d) + \tau(F_e) + \tau(F_g) \quad (10)$$

Because $\tau(F_g) > 0$, from Eq. (10) we have

$$\tau(f) > \tau(F_d) + \tau(F_e) \quad (11)$$

Let x_i be 1 in Eq. (9). Because Eq. (9) represents f_{i1}, we have

$$\tau(f_{i1}) \leq \tau(F_d) + \tau(F_e) \quad (12)$$

From Eqs. (11) and (12), we have $\tau(f_{i1}) < \tau(f)$. However, this contradicts the assumption of the lemma. In other words, if we assume that the MESOP has the form (9), then we have the contradiction. Q.E.D.

Lemma 5. Suppose that f is represented as $f = \bar{x}_i \cdot f_{i0} \oplus x_i \cdot f_{i1}$, where $f_{i0} = f(x_i = 0)$, $f_{i1} = f(x_i = 1)$. If $\tau(f) = \tau(f_{i0})$. Then a MESOP for f has a form $\bar{x}_i \cdot F_h \oplus F_k$, where F_h and F_k are ESOPs not containing the variable x_i.

Proof. Similar to Lemma 4. Q.E.D.

Lemma 6. Suppose that f is represented as $f = \bar{x}_i \cdot f_{i0} \oplus x_i \cdot f_{i1}$, where $f_{i0} = f(x_i = 0)$, $f_{i1} = f(x_i = 1)$ and $f_{i2} = f_{i0} \oplus f_{i1}$. If $\tau(f) = \tau(f_{i2})$, then a MESOP for f has a form $x_i \cdot F_p \oplus x_i \cdot F_q$, where F_p and F_q are ESOPs not containing the variable x_i.

Proof. Suppose that a MESOP for f is in the form:

$$\bar{x}_i \cdot F_p \oplus x_i \cdot F_q \oplus F_r \quad (13)$$

where $F_r \neq 0$. Then, we have

$$\tau(f) = \tau(F_p) + \tau(F_q) + \tau(F_r) \quad (14)$$

Because $\tau(F_r) > 0$, from Eq. (14) we have

$$\tau(f) > \tau(F_p) + \tau(F_q) \quad (15)$$

By setting x_i to 0 in Eq. (13), we have

$$F_p \oplus F_r \quad (16)$$

Equation (16) represents f_{i0}. Similarly, by setting x_i to 1 in Eq. (13), we have

$$F_q \oplus F_r \tag{17}$$

Equation (17) represents f_{i1}. By Eqs. (16) ⊕ (17), we have

$$F_p \oplus F_q \tag{18}$$

Because Eq. (18) represents f_{i2}, we have

$$\tau(f_{i2}) \leq \tau(F_p) + \tau(F_q) \tag{19}$$

From Eqs. (15) and (19), we have $\tau(f_{i2}) < \tau(f)$. However, this contradicts the assumption of the lemma. Therefore, we have the lemma. Q.E.D.

Lemma 7. For a given function f, let A and B be the values defined in Theorems 3 and 4. Then, $\tau(f) = A \Leftrightarrow A = B$.

Proof for $\Rightarrow$. From Theorem 3, A can be represented as either $\tau(f_{i1})$, $\tau(f_{i0})$ or $\tau(f_{i2})$.

1) When $\tau(f_{i1}) = A$. From Lemma 4, a MESOP for f has a form:

$$x_i \cdot F_d \oplus F_e \tag{20}$$

Because f can be represented as $f = x_i \cdot f_{i2} \oplus f_{i0}$, F_d represents a function f_{i2} and F_e represents a function f_{i0}. Also note that both F_d and f_e are MESOPs for f_{i2} and f_{i0}, respectively. If not, Eq. (20) is not a MESOP. Therefore, $\tau(F_d) = \tau(f_{i2})$ and $\tau(F_e) = \tau(f_{i0})$. Thus we have $\tau(f) = \tau(F_d) + \tau(F_e) + \tau(f_{i2}) = \tau(f_{i0})$. By the definition of B, we have $\tau(f_{i2}) + \tau(f_{i0}) \geq B$. Therefore, $\tau(f) \geq B$. On the other hand, by Theorem 4, we have $\tau(f) \leq B$. So $\tau(f) = B$. Hence, $A = B$.

2) When $\tau(f_{i0}) = A$. Similar to 1 by using Lemma 5.

3) When $\tau(f_{i2}) = A$. Similar to 1 by using Lemma 6. From 1, 2, and 3 we have $A = B$.

Proof for $\Leftarrow$. When $A = B$. By Theorem 3, we have $\tau(f) \geq A$. By Theorem 4, we have $\tau(f) \leq B$. Hence, $\tau(f) = A$. Q.E.D.

Theorem 5. For a given function f, let A and B be the values defined in Theorems 3 and 4. If $A \neq B$, then $\tau(f) \geq A + 1$.

Proof. By Lemma 7, if $A \neq B$, then $\tau(f) \neq A$. By Theorem 3, we have $\tau(f) \geq A$. Hence the theorem. Q.E.D.

By Lemma 7, and Theorem 5, we have the following:

Collorary 1. For a given function f, let A and B be the values defined in Theorems 3 and 4. Then $\tau(f) \geq L4$, where $L4 = \begin{cases} A & (A = B) \\ A + 1 & (A \neq B). \end{cases}$

The next theorem is due to Prof. S. Iwata of Tokai Univeristy [19].

Theorem 6. $\tau(f) \geq L5$, where $L5 = \max_i[\{\tau(f_{i0}) + \tau(f_{i1}) + \tau(f_{i2})\}/2]$, $f = x_i \cdot f_{i0} \oplus x_i \cdot f_{i1}$, $f_{i0} = f(x_i = 0)$, $f_{i1} = f(x_i = 1)$, $f_{i2} = f_{i0} \oplus f_{i1}$ and $i = 1, 2, \cdots, n$.

Proof. Let a MESOP for f be $F = x_i \cdot F_{i0} \oplus x_i \cdot F_{i1} \oplus F_{i2}$. Because $f_{i0} = F_{i0} \oplus F_{i2}$, we have

$$\tau(f_{i0}) \leq \tau(F_{i0}) + \tau(F_{i2}) \tag{21}$$

Because $f_{i1} = F_{i1} \oplus F_{i2}$, we have

$$\tau(f_{i1}) \leq \tau(F_{i1}) + \tau(F_{i2}) \tag{22}$$

Because $f_{i2} = F_{i0}\ \omega\ F_{i1}$, we have

$$\tau(f_{i2}) \leq \tau(F_{i0}) + \tau(F_{i1}) \tag{23}$$

By Eqs. (21) + (22) + (23), we have

$$\tau(f_{i0}) + \tau(f_{i1}) + \tau(f_{i2}) \leq 2\{\tau(F_{i0}) + \tau(F_{i1}) + \tau(F_{i2})\} = 2\tau(f) \tag{24}$$

Equation (24) holds for all possible i, thus we have the theorem. Q.E.D.

$L1$ to $L5$ derive lower bounds on the number of the products in MESOPs for a given function. In the following, we compare these lower bounds.

Remark 2. From Lemma 1 and Theorem 3, we have $L4 \geq L1$.

Lemma 8. Let $L4$ and $L5$ be the values defined in Collorary 1 and Theorem 6, respectively. Then $L5 \geq L4$.

Proof. Let A and B be the values defined in Theorems 3 and 4, respectively. From Theorem 3, A can be represented as either $\tau(f_{i0})$, $\tau(f_{i1})$ or $\tau(f_{i2})$, where $i = 1, 2, \cdots, n$.

(a) When $A = \tau(f_{i0})$.

From the definition of $L5$, we have $L5 \geq \{\tau(f_{i0}) + \tau(f_{i1}) + \tau(f_{i2})\}/2$. Also, from the definition of B, we have $B \leq \tau(f_{i1}) + \tau(f_{i2})$.

1) When $A = B$.

From Collorary 1, $L4 = A$. Therefore, we have

$$L5 \geqq \{\tau(f_{i0}) + \tau(f_{i1}) + \tau(f_{i2})\}/2 \geqq (A+B)/2 = A = L4 \quad (25)$$

2) When $A < B$.

From Collorary 1, $L4 = A + 1$. Therefore, we have $L5 \geq \{\tau(f_{i0}) + \tau(f_{i1}) + \tau(f_{i2})\}/2 \geq (A + B)/2 > A$. Because $L5$ is an integer, we have

$$L5 \geq A + 1 = L4 \quad (26)$$

From Eqs. (25) and (26), we have the lemma.

(b) When $A = \tau(f_{i1})$ or $A = \tau(f_{i2})$.

Similar to (a).

From (a) and (b), we have the lemma Q.E.D.

From Remark 2 and Lemma 8, $L1$ and $L4$ are unnecessary.

Definition 6. A *minterm* is a logical product containing a literal for each variable. A minterm implying a function is called a *minterm of f*. The set of minterms of f is denoted by $M(f)$. The number of elements in $M(f)$ is denoted by $|f|$.

Definition 7. For a given n-variable function, let $v \in M(f)$, and $E(v)$ be the set of the products covering v. The number of elements in $E(v)$ is 2^n.

Lemma 9. For a given n-variable function, $v \in M(f)$ and $q_i \in E(v)$, where i 1, 2, $\cdots$, 2^n. Then, $\tau(f) = 1 + \min_i\{\tau(f \oplus q_i)\}$.

Proof. Let F be an ESOP for f. F contains at least one of the products in $E(v)$. Let this product be q_i. Let G be the ESOP which is obtained by deleting the product q_i from F. Then we have $F = q_i \oplus G$. From this, we have $\tau(f) \leq \tau(F) = 1 + \tau(G)$. Because G represents the function $q_i \oplus f$, we have

$$\tau(f) \leq 1 + \tau(q_i \oplus f) \quad (27)$$

Because this relation holds for all possible i, we have

$$\tau(f) \leq 1 + \min_i\{\tau(q_i \oplus f)\} \quad (28)$$

Next, let H be a MESOP for f; H contains at least one of the products in $E(v)$. Let it be q_j. In this case, q_j is selected so that the number of the products in the MESOP for the function $f \oplus q_j$, which is obtained by deleting q_j from H, is minimum. Therefore, we have

$$\tau(f) \geq 1 + \min_j\{\tau(f \oplus q_j)\} \quad (29)$$

From Eqs. (28) and (29), we have the lemma. Q.E.D.

Definition 8. $Lmax(f) = \max\{L2, L3, L5\}$, where $L2$, $L3$ and $L5$ are the values defined in Lemma 2, Lemma 3 and Theorem 6, respectively.

Theorem 7. For a given n-variable function f let $v_i \in M(f)$ and $q_{ij} \in E(v_i)$, where $i = 1, 2, \cdots, 2^n$. Then $\tau(f) \geq LB$, where $LB = 1 + \max_i[\min_j\{Lmax(q_{ij} \oplus f)\}]$.

Proof. From Lemma 9, we have $\tau(f) = 1 + \min_j\{\tau(q_i \oplus f)\}$. Also from Definition 8, we have $\tau(q_{ij} \oplus f) \geq Lmax(q_{ij} \oplus f)$. Therefore, we have

$$\tau(f) \geq 1 + \min_j\{Lmax(q_{ij} \oplus f)\} \quad (30)$$

Because Eq. (30) holds for all possible i, we have the lemma. Q.E.D.

Next, we will compare the maximum values of $L2$, $L3$, $L5$ and LB.

Definition 9. For all the n-variable functions, the maximum values of $L2$, $L3$, $L5$, and LB are denoted by $L2max(n)$, $L3max(n)$, $L5max(n)$ and $LBmax(n)$, respectively.

Definition 10. The maximum number of products in MESOPs for n-variable functions is denoted by $\Psi(n)$.

Lemma 10. For a given n-variable function f, $L2max(n) \leq 2 \cdot \Psi(n-2)$, $L3max(n) \leq 3 \cdot \Psi(n-3)$, $L5max(n) \leq (3/2) \cdot \Psi(n-1)$ and $LBmax(n) \leq \max\{L2max(n), L3max(n), L5max(n)\} + 1$.

Table 1. Comparison of the maximum values of the lower bounds

Number of products	Maximum value of lower bounds			
	L2	L3	L5	LB
4	4	3	5	6
5	6	6	9	10
6	12	9	≤ 14	≤ 15
7	≤ 18	18	≤ 24	≤ 25
≥ 8	$\leq 2^{n-3}$	$\leq 3/4 \cdot 2^{n-3}$	$\leq 3/2 \cdot 2^{n-3}$	$\leq 3/2 \cdot 2^{n-3}+1$

Proof. It is trivial by the definitions of $L2$, $L3$ and $L5$ and Theorem 7. Q.E.D.

Because $\Psi(1) = 1$, $\Psi(2) = 2$, $\Psi(3) = 3$, $\Psi(4) = 6$, $\Psi(5) \leq 9$ and $\Psi(n)$ $2n$-$2(n \geq 6)$ [12], Table 1 compares the maximum values of the lower bounds.

4. Algorithm for Simplification

At present, as for the algorithms to obtain a MESOP for a given function, only exhaustive or virtually exhaustive methods are known. As for near-minimum ESOPs, several heuristic algorithms have been developed. All these algorithms simplify ESOPs by iterative improvements, and do not guarantee the minimality of the solutions [2-4]. By using the lower bound of a MESOP for a given function, an algorithm which proves the minimality of the simplified ESOPs is obtained. To obtain lower bounds for n-variable functions, the MESOPs for $(n - 1)$-variable functions are necessary. Up to now, all the MESOPs for the functions of four or fewer variables have been obtained [10]. The algorithm for the simplification of ESOPs for five-variable functions is as follows.

Simplification algorithm for five-variable functions.

1. Let LB be the lower bound on the number of products in ESOPs for a given function.

2. Expand the given function into one of the following forms: $f = \bar{x}_i \cdot f_{i0} \oplus x_i \cdot f_{i1}$, $f = x_i \cdot f_{i2} \oplus f_{i0}$, $f = \bar{x}_i \cdot f_{i2} \oplus f_{i2} \oplus f_{i1}$, where $f_{i2} = f_{i0} \oplus f_{i1}$.

3. Obtain the MESOPs for the subfunctions f_{i0}, f_{i1} and f_{i2} by using the table of the MESOPs for four-variable functions.

4. Obtain an ESOP for f by combining two MESOPs.

5. Simplify this ESOP by a heuristic minimization program EXMIN2 [14]. Let τ_a be the number of the products.

6. If $LB = \tau_a$, then the simplified ESOP is the minimum. Therefore, stop the algorithm. Otherwise, go to Step 2 and do Step 3 to Step 6 for another expansion. If all the expansions are exhausted (15 possible expansions) and still $LB \neq \tau_a$, then the minimality is not proved. In this case the ESOP with the fewest products ever found will be a near minimum solution.

5. Experimental Results and Discussions

For the five-variable functions, we cannot use the exhaustive method because the number of combinations to consider is too large. The set of n-variable functions can be partitioned into equivalence classes by various equivalence relations [8]. The NP equivalence relation, which is based on the negations and permutations of the inputs, partitions all the five-variable functions into about 1.2×10^6 equivalence classes [9]. The LP equivalence relation, which is based on the permutations of the inputs and the transformations of the literals, partitions all the five-variable functions into 6936 classes [12]. The number of LP equivalence classes is much smaller than that of the NP equivalence classes.

Therefore, in this paper, we use the LP equivalence relation to partition the five-variable functions. We simplified the representative functions of the LP equivalence classes of the five-variable functions by the algorithm is section 3, and obtained the number of products in the simplified ESOPs.

Table 2 shows the numbers of LP equivalence classes and the total number of functions requiring given number of products in the simplified ESOPs. Also, the numbers of the functions whose minimalities of the simplified ESOPs have been guaranteed by using the algorithm in section 4 are shown.

Table 3 compares the lower bounds and shows the number of equivalence classes whose lower bounds are distinct (i.e., larger than others) among $L2$, $L3$, and $L5$. From these tables, we can see the following:

1) an arbitrary five-variable function can be realized by an ESOP with at most nine products;

2) when the number of products in the MESOPs is at most 6, the algorithm proved the minimality of the

Table 2. Simplification of five-variable ESOPs

Number of products	Number of equivalence classes	Number of total functions	Number of equivalence classes which are proved to be minimum by each lower bound				Proved to be minimum		Rate of guarantee (%)
			L2	L3	L5	LB	No. of equivalence classes	Number of total functions	
0	1	1	1	1	1	1	1	1	100
1	1	243	1	1	1	1	1	243	100
2	4	24948	4	4	4	4	4	24948	100
3	19	1351836	19	19	19	19	19	1351836	100
4	137	39365190	136	96	137	137	137	39365190	100
5	971	545193342	886	339	966	971	971	545193342	100
6	3572	2398267764	1581	221	3204	3571	3571	2398252212	99.99
7	2143	1299295404	0	0	1124	2036	2036	1227138012	94.45
8	86	11460744	0	0	35	50	50	5718168	49.89
9	2	7824	0	0	1	1	1	48	0.61
Total	6936	4294967296	2627	680	5492	6791	6791	4217044000	98.19

Table 3. Lower bounds for five-variable LP equivalence representative functions

No. of products	L2	L3	L5	LB
1	1.00 0	1.00 0	1.00 0	1.00
2	2.00 0	2.00 0	2.00 0	2.00
3	3.00 0	3.00 0	3.00 0	3.00
4	3.99 0	3.70 0	4.00 0	4.00
5	4.91 0	4.35 0	4.99 1	5.00
6	5.44 52	5.02 4	5.90 1018	6.00
7	5.81 0	5.45 0	6.52 1063	6.95
8	5.92 0	5.92 0	7.40 35	7.58
9	6.00 0	6.00 0	8.50 2	8.50
Average value total	5.45 52	5.04 4	5.94 2119	6.12

Lower part of each column represents the number of equivalence classes whose lower bounds are different.

simplified ESOPs for almost 100 percent of the functions;

3) the algorithm prove the minimality of the ESOPs for about 98 percent of the functions. The remaining ESOPs could not prove the minimality, but we proved the minimality of these ESOPs by using another method [20];

4) *LB* is the greatest lower bound among four. For most functions, $L5 > L2 > L3$ holds, but there are some exceptions. Also, $L2$, $L3$ and $L5$ all are essential for the estimation of the lower bounds of functions.

In this algorithm, a heuristic minimization algorithm EXMIN2, which simplifies ESOPs by iterative improvements is used. As mentioned before, EXMIN2 does not prove the minimality of the simplified ESOPs.

6. Conclusions

In this paper, we derived lower bounds on the number of products in MESOPs for n-variable functions when the numbers of products in MESOPs for (n - 1)-variable functions are available. We developed a minimization algorithm for ESOPs of five-variable functions. The features of this algorithm are: 1) ability to obtain a lower bound; and 2) ability to stop the algorithm when the solution is proved to be minimum. Thus, the solutions are more reliable than those obtained by the existing heuristic algorithms.

This algorithm simplified five-variable ESOPs and proved the minimality for about 98 percent of all five-variable functions. Although various minimization algorithms for logic expressions have been developed, no algorithm used the minimized results of all the functions with fewer variables. When this algorithm is extended for the functions for six or more variables, we can obtain the initial ESOPs with fewer products, and simplify ESOPs in a shorter time. Logic minimization programs are indispensable tools for the design of VLSI circuits. The minimization algorithm proposed in this paper is useful for the design of compact and easily testable VLSI circuits. The extension of this algorithm to the multiple-valued input two-valued output functions [13, 18] is a future work.

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APPENDIX

⟨Appendix 1⟩

Proof for Lemma 1

Let F be a MESOP for f and be represented as follows:

$$F(0)x^0 \oplus F(1)x^1 \oplus F(2)x^2 \quad \text{(A1)}$$

where $F(a)(a \in \{0, 1, 2\})$ are ESOPs which do not contain the variable x. By setting $x = 0$ in (A1),

$$F(0) \oplus F(2) = f(0) \quad \text{(A2)}$$

By setting $x = 1$, in Eq. (A1),

$$F(1) \oplus F(2) = f(1) \quad \text{(A3)}$$

By Eqs. (A2) $\oplus$ (A3), we have

$$F(0) \oplus F(1) = f(0) \oplus f(1) \quad \text{(A4)}$$

Let $\tau(a) = \tau(F(a))$. From Eq. (A4), we have $\tau(0) + \tau(1) \geq \tau(f(0) \oplus f(1))$. Note than $\tau(f) = \tau(0) + \tau(1) + \tau(2)$. Because $\tau(2) \geq 0$, we have $\tau(f) \geq \tau(f(0) \oplus f(1))$. Q.E.D.

⟨Appendix 2⟩

Proof for Lemma 2

Let F be a MESOP for f and be represented as follows:

$$F(0,0)x^0y^0 \oplus F(0,1)x^0y^1 \oplus F(0,2)x^0y^2 \oplus F(1,0)x^1y^0 \oplus F(1,1)x^1y^1 \oplus F(1,2)x^1y^2 \oplus F(2,0)x^2y^0 \oplus F(2,1)x^2y^1 \oplus F(2,2)x^2y^2 \quad \text{(B1)}$$

where $F(a, b)(a, b \in \{0, 1, 2\})$ are ESOPs which do not contain the variable x or y. By setting $(x, y) = (0, 0)$ in Eq. (B1),

$$F(0,0) \oplus F(0,2) \oplus F(2,0) \oplus F(2,2) = f(0,0) \quad \text{(B2)}$$

By setting $(x, y) = (1, 1)$ in Eq. (B1),

$$F(1,1) \oplus F(1,2) \oplus F(2,1) \oplus F(2,2) = f(1,1) \quad \text{(B3)}$$

By setting $(x, y) = (0, 1)$ in Eq. (B1),

$$F(0,1) \oplus F(0,2) \oplus F(2,1) \oplus F(2,2) = f(0,1) \quad \text{(B4)}$$

By setting $(x, y) = (1, 0)$ in Eq. (B1),

$$F(1,0) \oplus F(1,2) \oplus F(2,0) \oplus F(2,2) = f(1,0) \quad \text{(B5)}$$

From Eqs. (B2) and (B4),

$$F(0,0) \oplus F(0,1) \oplus F(2,0) \oplus F(2,1) = f(0,0) \oplus f(0,1) \quad \text{(B6)}$$

From Eqs. (B2) and (B5),

$$F(0,0) \oplus F(0,2) \oplus F(1,0) \oplus F(1,2) = f(0,0) \oplus f(1,0) \quad \text{(B7)}$$

From Eqs. (B3) and (B4),

$$F(0,1) \oplus F(0,2) \oplus F(1,1) \oplus F(1,2) = f(1,1) \oplus f(0,1) \quad \text{(B8)}$$

From Eqs. (B3) and (B5),

$$F(1,0) \oplus F(1,1) \oplus F(2,0) \oplus F(2,1) = f(1,1) \oplus f(1,0) \quad \text{(B9)}$$

Let $\tau(a, b) = \tau(F(a, b))$. From Eqs. (B6) to (B9), we have $\tau(0, 0) + \tau(0, 1) + \tau(2, 0) + \tau(2, 1) \geq \tau(0, 0 : 0, 1)$, $\tau(0, 0 : 0, 1)$, $\tau(0, 0) + \tau(1, 0) + \tau(1, 2) \geq \tau(0, 0 : 1, 0)$, $\tau(0, 1) + \tau(0, 2) + \tau(1, 1) + \tau(1, 2) \geq \tau(1, 1 : 0, 1)$ and $\tau(1, 0) + \tau(1, 1) + \tau(2, 0) + \tau(2, 1) \geq \tau(1, 1 : 1, 0)$. By adding the four inequations in the foregoing, we have $2\{\tau(0, 0) + \tau(0, 1) + \tau(0, 2) + \tau(1, 0) + \tau(1, 1) + \tau(1, 2) + \tau(2, 0) + \tau(2, 1) \geq \tau(0, 0 : 0, 1) + \tau(0, 0 : 1, 0) + \tau(1, 1 : 0, 1) + \tau(1, 1, : 1, 0)$. Note that $\tau(f) = \tau(0, 0) + \tau(0, 1) + \tau(0, 2) + \tau(1, 0) + \tau(1, 1) + \tau(1, 2) + \tau(2, 0) + \tau(2, 1) + \tau(2, 2)$. Because $\tau(2, 2) \geq 0$, we have $2\tau(f) \geq \tau(0, 0 : 0, 1) + \tau(0, 0 : 1, 0) + \tau(1, 1 : 0, 1) + \tau(1, 1 : 1, 0)$. Hence the lemma. Q.E.D.

⟨Appendix 3⟩

Proof for Lemma 3

Let F be a MESOP for f and be represented as follows:

$$F(0,0,0)x^0y^0z^0 \oplus F(0,0,1)x^0y^0z^1 \oplus F(0,0,2)x^0y^0z^2 \oplus F(0,1,0)x^0y^1z^0 \oplus F(0,1,1)x^0y^1z^1 \oplus F(0,1,2)x^0y^1z^2 \oplus F(0,2,0)x^0y^2z^0 \oplus F(0,2,1)x^0y^2z^1 \oplus F(0,2,2)x^0y^2z^2 \oplus F(1,0,0)x^1y^0z^0 \oplus F(1,0,1)x^1y^0z^1 \oplus F(1,0,2)x^1y^0z^2 \oplus F(1,1,0)x^1y^1z^0 \oplus F(1,1,1)x^1y^1z^1 \oplus F(1,1,2)x^1y^1z^2 \oplus F(1,2,0)x^1y^2z^0 \oplus F(1,2,1)x^1y^2z^1 \oplus F(1,2,2)x^1y^2z^2 \oplus F(2,0,0)x^2y^0z^0$$

$$\oplus F(2,0,1)x^2y^0z^1 \oplus F(2,0,2)x^2y^0z^2 \oplus F(2,1,0)x^2y^1z^0 \oplus F(2,1,1)x^2y^1z^1 \oplus F(2,1,2)x^2y^1z^2 \oplus F(2,2,0)x^2y^2z^0 \oplus F(2,2,1)x^2y^2z^1 \oplus F(2,2,2)x^2y^2z^2 \quad \text{(C1)}$$

where $F(a, b, c)(a, b, c \in \{0, 1, 2\})$ are ESOPs which do not contain variable $x, y\ z$. By setting $(x, y, z) = (0, 0, 0)$ in Eq. (C1),

$$F(0,0,0) \oplus F(0,0,2) \oplus F(0,2,0) \oplus F(0,2,2) \oplus F(2,0,0) \oplus F(2,0,2) \oplus F(2,2,0) \oplus F(2,2,2) = f(0,0,0) \quad \text{(C2)}$$

By setting $(x, y, z) = (0, 1, 1)$ in Eq. (C1),

$$F(0,1,1) \oplus F(0,1,2) \oplus F(0,2,1) \oplus F(0,2,2) \oplus F(2,1,1) \oplus F(2,1,2) \oplus F(2,2,1) \oplus F(2,2,2) = f(0,1,1) \quad \text{(C3)}$$

By setting $(x, y, z) = (1, 0, 1)$ in Eq. (C1),

$$F(1,0,1) \oplus F(1,0,2) \oplus F(1,2,1) \oplus F(1,2,2) \oplus F(2,0,1) \oplus F(2,0,2) \oplus F(2,2,1) \oplus F(2,2,2) = f(1,0,1) \quad \text{(C4)}$$

By setting $(x, y, z) = (1, 1, 0)$ in Eq. (C1),

$$F(1,1,0) \oplus F(1,1,2) \oplus F(1,2,0) \oplus F(1,2,2) \oplus F(2,1,0) \oplus F(2,1,2) \oplus F(2,2,0) \oplus F(2,2,2) = f(1,1,0) \quad \text{(C5)}$$

By setting $(x, y, z) = (0, 0, 1)$ in Eq. (C1),

$$F(0,0,1) \oplus F(0,0,2) \oplus F(0,2,1) \oplus F(0,2,2) \oplus F(2,0,1) \oplus F(2,0,2) \oplus F(2,2,1) \oplus F(2,2,2) = f(0,0,1) \quad \text{(C6)}$$

By setting $(x, y, z) = (0, 1, 0)$ in Eq. (C1),

$$F(0,1,0) \oplus F(0,1,2) \oplus F(0,2,0) \oplus F(0,2,2) \oplus F(2,1,0) \oplus F(2,1,2) \oplus F(2,2,0) \oplus F(2,2,2) = f(0,1,0) \quad \text{(C7)}$$

By setting $(x, y, z) = (1, 0, 0)$ in Eq. (C1),

$$F(1,0,0) \oplus F(1,0,2) \oplus F(1,2,0) \oplus F(1,2,2) \oplus F(2,0,0) \oplus F(2,0,2) \oplus F(2,2,0) \oplus F(2,2,2) = f(1,0,0) \quad \text{(C8)}$$

By setting $(x, y, z) = (1, 1, 1)$ in Eq. (C1),

$$F(1,1,1) \oplus F(1,1,2) \oplus F(1,2,1) \oplus F(1,2,2) \oplus F(2,1,1) \oplus F(2,1,2) \oplus F(2,2,1) \oplus F(2,2,2) = f(1,1,1) \quad \text{(C9)}$$

By (C2) ⊕ (C6),

$$F(0,0,0) \oplus F(0,0,1) \oplus F(0,2,0) \oplus F(0,2,1) \oplus F(2,0,0) \oplus F(2,0,1) \oplus F(2,2,0) \oplus F(2,2,1) = f(0,0,0) \oplus f(0,0,1) \quad \text{(C10)}$$

By Eqs. (C2) ⊕ (C7),

$$F(0,0,0) \oplus F(0,0,2) \oplus F(0,1,0) \oplus F(0,1,2) \oplus F(2,0,0) \oplus F(2,0,2) \oplus F(2,1,0) \oplus F(2,1,2) = f(0,0,0) \oplus f(0,1,0) \quad \text{(C11)}$$

By Eqs. (C2) ⊕ (C8),

$$F(0,0,0) \oplus F(0,0,2) \oplus F(0,2,0) \oplus F(0,2,2) \oplus F(1,0,0) \oplus F(1,0,2) \oplus F(1,2,0) \oplus F(1,2,2) = f(0,0,0) \oplus f(1,0,0) \quad \text{(C12)}$$

By Eqs. (C3) ⊕ (C6),

$$F(0,0,1) \oplus F(0,0,2) \oplus F(0,1,1) \oplus F(0,1,2) \oplus F(2,0,1) \oplus F(2,0,2) \oplus F(2,1,1) \oplus F(2,1,2) = f(0,1,1) \oplus f(0,0,1) \quad \text{(C13)}$$

By Eqs. (C3) ⊕ (C7),

$$F(0,1,0) \oplus F(0,1,1) \oplus F(0,2,0) \oplus F(0,2,1) \oplus F(2,1,0) \oplus F(2,1,1) \oplus F(2,2,0) \oplus F(2,2,1) = f(0,1,1) \oplus f(0,1,0) \quad \text{(C14)}$$

By Eqs. (C3) ⊕ (C9),

$$F(0,1,1) \oplus F(0,1,2) \oplus F(0,2,1) \oplus F(0,2,2) \oplus F(1,1,1) \oplus F(1,1,2) \oplus F(1,2,1) \oplus F(1,2,2) = f(0,1,1) \oplus f(1,1,1) \quad \text{(C15)}$$

By Eqs. (C4) ⊕ (C6),

$$F(0,0,1) \oplus F(0,0,2) \oplus F(0,2,1) \oplus F(0,2,2) \oplus F(1,0,1) \oplus F(1,0,2) \oplus F(1,2,1) \oplus F(1,2,2) = f(1,0,1) \oplus f(0,0,1) \quad \text{(C16)}$$

By Eqs. (C4) ⊕ (C8),

$$F(1,0,0) \oplus F(1,0,1) \oplus F(1,2,0) \oplus F(1,2,1) \oplus F(2,0,0) \oplus F(2,0,1) \oplus F(2,2,0) \oplus F(2,2,1) = f(1,0,1) \oplus f(1,0,0) \quad \text{(C17)}$$

By Eqs. (C4) ⊕ (C9),

$$F(1,0,1) \oplus F(1,0,2) \oplus F(1,1,1) \oplus F(1,1,2) \oplus F(2,0,1) \oplus F(2,0,2) \oplus F(2,1,1) \oplus F(2,1,2) = f(1,0,1) \oplus f(1,1,1) \quad \text{(C18)}$$

By Eqs. (C5) ⊕ (C7),

$$F(0,1,0) \oplus F(0,1,2) \oplus F(0,2,0) \oplus F(0,2,2) \oplus F(1,1,0) \oplus F(1,1,2) \oplus F(1,2,0) \oplus F(1,2,2) = f(1,1,0) \oplus f(0,1,0) \quad \text{(C19)}$$

By Eqs. (C5) ⊕ (C8),

$$F(1,0,0) \oplus F(1,0,2) \oplus F(1,1,0) \oplus F(1,1,2) \oplus F(2,0,0) \oplus F(2,0,2) \oplus F(2,1,0) \oplus F(2,1,2) = f(1,1,0) \oplus f(1,0,0) \quad \text{(C20)}$$

By Eqs. (C5) ⊕ (C9),

$$F(1,1,0) \oplus F(1,1,1) \oplus F(1,2,0) \oplus F(1,2,1) \oplus F(2,1,0) \oplus F(2,1,1) \oplus F(2,2,0) \oplus F(2,2,1) = f(1,1,0) \oplus f(1,1,1) \quad \text{(C21)}$$

Let $\tau(a, b, c) = \tau(F(a, b, c))$ and $\tau(a, b, c : d, e, h) = \tau(f(a, b, c) \oplus f(d, e, h))$.

From Eqs. (C10) to (C21), we have

$$\begin{aligned}
&3\tau(0,0,0)+3\tau(0,0,1)+4\tau(0,0,2)+3\tau(0,1,0)\\
&+3\tau(0,1,1)+4\tau(0,1,2)+4\tau(0,2,0)+4\tau(0,2,1)\\
&+4\tau(0,2,2)+3\tau(1,0,0)+3\tau(1,0,1)+4\tau(1,0,2)\\
&+3\tau(1,1,0)+3\tau(1,1,1)+4\tau(1,1,2)+4\tau(1,2,0)\\
&+4\tau(1,2,1)+4\tau(1,2,2)+4\tau(2,0,0)+4\tau(2,0,1)\\
&+4\tau(2,0,2)+4\tau(2,1,0)+4\tau(2,1,1)+4\tau(2,1,2)\\
&+4\tau(2,2,0)+4\tau(2,2,1) \geq\\
&\{\tau(0,0,0:0,0,1)+\tau(0,0,0:0,1,0)\\
&+\tau(0,0,0:1,0,0)+\tau(0,1,1:0,0,1)\\
&+\tau(0,1,1:0,1,0)+\tau(0,1,1:1,1,1)\\
&+\tau(1,0,1:0,0,1)+\tau(1,0,1:1,0,0)\\
&+\tau(1,0,1:1,1,1)+\tau(1,1,0:0,1,0)\\
&+\tau(1,1,0:1,0,0)+\tau(1,1,0:1,1,1)\}
\end{aligned}$$

Note that $\tau(f) = \tau(0, 0, 0) + \tau(0, 0, 1) + \tau(0, 0, 2) + \tau(0, 1, 0) + \tau(0, 1, 1) + \tau(0, 1, 2) + \tau(0, 2, 1) + \tau(0, 2, 2) + \tau(1, 0, 0) + \tau(1, 0, 1) + \tau(1, 0, 2) + \tau(1, 1, 0) + \tau(1, 1, 1) + \tau(1, 1, 2) + \tau(1, 2, 0) + \tau(1, 2, 1) + \tau(1, 2, 2) + \tau(2, 0, 0) + \tau(2, 0, 1) + \tau(2, 0, 2) + \tau(2, 1, 0) + \tau(2, 1, 1) + \tau(2, 1, 2), + \tau(2, 2, 0) + \tau(2, 2, 1) + \tau(2, 2, 2)$.

Because $\tau(a, b, c) \geq 0$, we have

$$\begin{aligned}
4\cdot\tau(f) \geq &\{\tau(0,0,0:0,0,1)+\tau(0,0,0:0,1,0)\\
&+\tau(0,0,0:1,0,0)+\tau(0,1,1:0,0,1)\\
&+\tau(0,1,1:0,1,0)+\tau(0,1,1:1,1,1)\\
&+\tau(1,0,1:0,0,1)+\tau(1,0,1:1,0,0)\\
&+\tau(1,0,1:1,1,1)+\tau(1,1,0:0,1,0)\\
&+\tau(1,1,0:1,0,0)+\tau(1,1,0:1,1,1)\}
\end{aligned}$$

Hence the lemma. Q.E.D.

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