# On the Number of Dependent Variables for Incompletely Specified Multiple-Valued Functions

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**Abstract**— This paper considers the minimization of dependent variables in functions with many don't cares. It also derives the conditions for almost all randomly generated function to be redundant in at least one variable. Experimental results support the validity of the approach.

## I Introduction

Logic minimization often means to reduce the number of products to represent the given function [2]. However, in the case of incompletely specified functions (i.e., functions with don't cares), at least two problems exist [7]: The first one is to reduce the number of the products to represent the function, and the second one is to reduce the number of dependent variables. The first problem is important when the given function is realized as a sum-of-products expression (SOP) [2], an EXOR sum-of-products expressions (ESOP) [11], etc. The second problem is important when the given function is realized as a ROM or a RAM, since the number of the input variables only determines the cost of the realization.

**Example 1.1** *Consider the four-variable logic function shown in Fig. 1.1, where the blank cells denote don't cares. The SOP with the minimum number of products is*

$$\mathcal{F}_1 = x_1x_4 \vee x_2\bar{x}_3,$$

*while the SOP with the minimum number of dependent variables is*

$$\mathcal{F}_2 = x_1x_2 \vee x_1x_4 \vee x_2x_4.$$

*Note that $\mathcal{F}_1$ has two products and depends on four variables, while $\mathcal{F}_2$ has three products and depends on only three variables.* *(End of Example)*

As shown in this example, the minimization of the number of products is different from the minimization of the number of dependent variables.

In this paper, we will consider the minimization of the number of dependent variables. The rest of the paper is organized as follows:

In Section II, we will give definitions and basic properties.

In Section III, we will formulate the minimization problem of the dependent variables in the function $f$ :

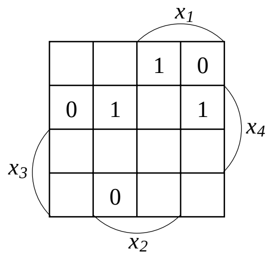


Figure 1.1: Four-variable function with don't cares.

$P^n \to Q$, where $P = \{0, 1, \ldots, p-1\}$, $Q = \{0, 1, \ldots, q-1\}$. And we will show the minimization algorithm.

In Section IV, we will consider random functions $f : P^n \to Q$, and derive formulas to predict the number of redundant variables when the percentage of specified minterms is very small. For example, when $p = 2$, $q = 2$, $n = 17$, and 512 minterms are mapped to zeros, 512 minterms are mapped to ones, and other minterms are mapped to don't cares, the formula predicts that at least one variable is redundant in 91.6% of the functions. In the area of knowledge engineering, more than 99% of the entries are don't cares [6, 8, 9], and this formula is useful to predict the number of dependent variables.

In Section V, we will compare the analysis with experimental results.

## II Definitions and Basic Properties

**Definition 2.1** *An* **incompletely specified multiple-valued function** *(function for short) $f$ is a mapping $D \to Q$, where $D \subset P^n$, $P = \{0, 1, \ldots, p-1\}$, and $Q = \{0, 1, \ldots, q-1\}$.*

**Definition 2.2** *An incompletely specified multiple-valued function is represented by a set of* **characteristic functions** *$F_i$, where $F_i(\boldsymbol{a}) = 1$ iff $f(\boldsymbol{a}) = i$ $(i = 0, 1, \ldots, q-1)$. Note that $F_iF_j = 0$ $(i \neq j)$. If $\boldsymbol{a} \in P^n - D$, then the value of $f(\boldsymbol{a})$ is unspecified, and is denoted by $d$ (don't care).*

**Definition 2.3** *[12] Two-valued variables are often represented by $x_i$ $(i = 1, 2, \ldots, n)$. Multiple-valued variables are represented by $X_i$ $(i = 1, 2, \ldots, n)$. $X_i$*

Table 2.1:

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $f$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 |

*takes one of values in $P$. A literal is defined as $X_i^S = 1$ if $X_i \in S$, and $X_i^S = 0$ otherwise, where $S \subseteq P$. For two-valued case, $X^0 = \bar{x}$, $X^1 = x$, and $X^{\{0,1\}} = 1$. A product of literals is a* **product term***, and a sum of products is a* **sum-of-products expression** *(SOP). If all the products are prime and no product can be removed from the SOP without changing the function, then it is an* **irredundant sum-of-products expression** *(ISOP).*

**Example 2.1** *Consider the function in Example 1.1. In this case, $p = q = 2$ and $n = 4$. Table 2.1 also shows this function. The characteristic functions are*

$$F_0 = x_1^0 x_2^0 x_3^0 x_4^1 \vee x_1^0 x_2^1 x_3^1 x_4^0 \vee x_1^1 x_2^0 x_3^0 x_4^0 \text{ and}$$
$$F_1 = x_1^0 x_2^1 x_3^0 x_4^1 \vee x_1^1 x_2^0 x_3^0 x_4^1 \vee x_1^1 x_2^1 x_3^0 x_4^0.$$

*(End of Example)*

**Example 2.2** *Consider the function in Table 2.2. In this case, $p = 3$, $q = 4$ and $n = 4$. The characteristic functions are*

$$F_0 = X_1^0 X_2^2 X_3^1 X_4^2 \vee X_1^2 X_2^1 X_3^1 X_4^0,$$
$$F_1 = X_1^0 X_2^1 X_3^1 X_4^2 \vee X_1^2 X_2^1 X_3^2 X_4^1,$$
$$F_2 = X_1^0 X_2^1 X_3^2 X_4^1 \vee X_1^2 X_2^2 X_3^2 X_4^2, \text{ and}$$
$$F_3 = X_1^0 X_2^0 X_3^1 X_4^2 \vee X_1^1 X_2^1 X_3^1 X_4^2.$$

*(End of Example)*

**Lemma 2.1** *Let $f$ be a function $P^n \rightarrow Q$. $f$ is expanded with respect to $X_1$ as follows:*

$$f = X_1^0 f_0 \vee X_1^1 f_1 \vee \cdots \vee X_1^{p-1} f_{p-1},$$

*where $f_i = f(|X_1 = i)$.*

**Definition 2.4** *$f$* **depends on** *$X_i$ if there exists a pair of vectors*

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_i, \ldots, a_n) \text{ and}$$
$$\boldsymbol{b} = (a_1, a_2, \ldots, b_i, \ldots, a_n),$$

*such that both $f(\boldsymbol{a})$ and $f(\boldsymbol{b})$ are specified, and $f(\boldsymbol{a}) \neq f(\boldsymbol{b})$.*

Table 2.2:

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $f$ |
|---|---|---|---|---|
| 0 | 2 | 1 | 2 | 0 |
| 2 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 2 | 1 |
| 2 | 1 | 2 | 1 | 1 |
| 0 | 1 | 2 | 1 | 2 |
| 2 | 2 | 2 | 2 | 2 |
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 1 | 2 | 3 |

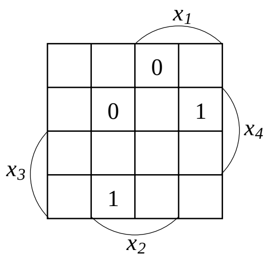


Figure 2.1: Four-variable function with don't cares.

If $f$ depends on $X_i$, then $X_i$ is **essential** in $f$, and $X_i$ must appear in any expression for $f$.

**Definition 2.5** *Two functions $f$ and $g : P^n \rightarrow Q$ are* **compatible** *when the following condition holds: For any $\boldsymbol{a} \in P^n$, if both $f(\boldsymbol{a})$ and $g(\boldsymbol{a})$ are specified, then $f(\boldsymbol{a}) = g(\boldsymbol{a})$.*

**Definition 2.6** *$X_1$ is* **non-essential** *in $f$ iff $f_0$, $f_1$, $\ldots$, and $f_{p-1}$ are compatible each other.*

**Theorem 2.1** *If $X_1$ is non-essential in $f$, then $f$ can be represented by an expression without $X_1$.*

**Definition 2.7** *A set of variables $\{X_i\}$ is* **redundant** *if $f$ can be represented without using $\{X_i\}$.*

**Example 2.3** *Consider the function $f$ in Fig. 2.1. It is easy to verify that all the variables are non-essential. Note that $f$ can be represented as $\mathcal{F}_1 = \bar{x}_2 \vee x_3$ or $\mathcal{F}_2 = x_1 \oplus \bar{x}_4$. The redundant sets of variables are $\{x_1, x_4\}$ in $\mathcal{F}_1$ and $\{x_2, x_3\}$ in $\mathcal{F}_2$. (End of Example)*

Essential variables must appear in every expression for $f$, while non-essential variables, may appear in some expressions and not in others.

**Example 2.4** *Consider the function $f$ in Table 2.2. Since $f(0,1,2,1) = 2$ and $f(2,1,2,1) = 1$, $f$ depends on $X_1$, and $X_1$ is essential. Since $f(0,0,1,2) = 3$ and*

*$f(0,2,1,2)=0$, $f$ depends on $X_2$, and $X_2$ is essential. However, $X_3$ and $X_4$ are non-essential. To represent $f$, both $X_1$ and $X_2$ are necessary. In addition, either $X_3$ or $X_4$ is necessary. This fact will be shown in Example 3.2.* *(End of Example)*

## III Minimization of Dependent Variables

In this section, we will consider the problem to represent a given function by expressions with the minimum number of variables. For two-valued logic functions, this problem has been considered by Halatsis-Gaitanis [5], Brown [1], and others [7, 3, 4]. Here, we will consider the cases of multiple-valued functions $P^n \to Q$, where $P=\{0,1,\ldots,p-1\}$ and $Q=\{0,1,\ldots,q-1\}$. The following is an extension of Halatsis-Gaitanis's method.

**Algorithm 3.1** *(Minimization of Dependent Variables)*

1) *Express the characteristic functions $F_i$ $(i = 0,1,\ldots,q-1)$ by SOPs:*

$$F_i = \bigvee_{k=1}^{t(F_i)} r(i,k),$$

*where $t(F_i)$ denotes the number of products in the SOP for $F_i$, and $r(i,k)$ denotes the $k$-th product in $F_i$.*

2) *For each pair of the characteristic functions, $F_i$ and $F_j$ $(i \neq j)$, do the following:*

*For each pair of products $[r(i,k), r(j,\ell)]$, associate a sum-of-literals $s(i,j,k,\ell)$ defined by*

$$s(i,j,k,\ell) = \bigvee_{m=1}^{n} y_m,$$

*where $y_m = 0$ if $S_m \cap T_m \neq 0$, $y_m = x_m$ if $S_m \cap T_m = 0$, $r(i,k) = X_1^{S_1}X_2^{S_2}\cdots X_n^{S_n}$, $r(j,\ell) = X_1^{T_1}X_2^{T_2}\cdots X_n^{T_n}$, and $m=1,2,\ldots,n$.*

3) *Define a Boolean function*

$$R = \bigwedge_{i=0}^{q-2}\ \bigwedge_{j=i+1}^{q-1}\ \bigwedge_{k=1}^{t(F_i)}\ \bigwedge_{\ell=1}^{t(F_j)} s(i,j,k,\ell).$$

4) *Represent $R$ as an ISOP. The product with the minimal number of literals corresponds to the set of minimal dependent variables.*

(Correctness of the Algorithm)
Consider two characteristic functions $F_i$ and $F_j$, where $i \neq j$. Let $F_i$ have a product $c_i = X_1^{S_1}X_2^{S_2}\cdots X_n^{S_n}$, and $F_j$ have a product $c_j = X_1^{T_1}X_2^{T_2}\cdots X_n^{T_n}$. Let $I = \{1,2,\ldots,n\}$.

1) By definition of the characteristic functions, there exists at least one variable $x_m$ such that $S_m \cap T_m = \phi$, and $m \in I$. If there is no such $m$, then $S_m \cap T_m \neq 0$ for all $m \in I$. So, there exists a vector $\boldsymbol{a}$ such that $F_i(\boldsymbol{a}) = F_j(\boldsymbol{a}) = 1$. This contradicts the definition of the characteristic functions.

2) Let $L \subseteq I$ and, for all $m \in L$, $S_m \cap T_m = \phi$, where $c_i = X_1^{S_1}X_2^{S_2}\cdots X_n^{S_n}$ and $c_j = X_1^{T_1}X_2^{T_2}\cdots X_n^{T_n}$. Without loss of generality, assume that $L = \{1,2,3,\ldots,k\}$ $(1 \le k \le n)$. In this case, we claim that at least one variable in $\{x_1,x_2,\ldots,x_k\}$ is necessary to distinguish $c_i$ from $c_j$. This corresponds to $s(i,j,k,\ell)$ in the algorithm.

3) This condition must hold for all the pair of the products $(c_i, c_j)$ for different characteristic functions $F_i$ and $F_j$. Thus, the Boolean function $R$ in the algorithm represents the condition.

4) Therefore, each product of the ISOP for $R$ represents the condition necessary to distinguish $F_i$ from $F_j$.

5) Since $s(i,j,k,\ell)$ shows the minimum condition to distinguish $F_i$ from $F_j$, each product in the ISOP corresponds to the set of minimal dependent variables. (End of the Correctness)

**Example 3.1** *Consider the function in Example 1.1.*

1) *SOPs for the characteristic functions are given. Each product is labeled as follows:*

$$\begin{aligned}
r(0,1)&=x_1^0x_2^0x_3^0x_4^1,\\
r(0,2)&=x_1^0x_2^1x_3^1x_4^0,\\
r(0,3)&=x_1^1x_2^0x_3^0x_4^0,\\
r(1,1)&=x_1^0x_2^1x_3^0x_4^1,\\
r(1,2)&=x_1^1x_2^0x_3^0x_4^1,\\
r(1,3)&=x_1^1x_2^1x_3^0x_4^0.
\end{aligned}$$

2) *For each pair of products, find the variables whose products are null. Note that in $r(0,1)$ and $r(1,1)$, only the product is null in $x_2$. So, we have*

$$s(0,1,1,1) = x_2.$$

*Similarly, we have*

$$\begin{aligned}
s(0,1,1,2)&=x_1,\\
s(0,1,1,3)&=x_1 \vee x_2 \vee x_4,\\
s(0,1,2,1)&=x_3 \vee x_4.
\end{aligned}$$

*For the pair of $r(0,2)$ and $r(1,2)$, the products are null in all variables. So, we have*

$$s(0,1,2,2) = x_1 \vee x_2 \vee x_3 \vee x_4.$$

*Similarly, we have*

$$s(0,1,2,3)=x_1 \vee x_3,$$
$$s(0,1,3,1)=x_1 \vee x_2 \vee x_4,$$
$$s(0,1,3,2)=x_4,$$
$$s(0,1,3,3)=x_2.$$

3) *$R$ is obtained as the logical product of $s(i,j,k,\ell)$, and we have*

$$R=x_2 x_1 (x_1 \vee x_2 \vee x_4)(x_3 \vee x_4)(x_1 \vee x_2 \vee x_3 \vee x_4)(x_1 \vee x_3)(x_1 \vee x_2 \vee x_4) x_4 x_2 = x_1 x_2 x_4.$$

*This means that the function can be represented by the SOP with variable set $\{x_1, x_2, x_4\}$:*

$$\mathcal{F} = x_1^0 x_2^1 x_4^1 \vee x_1^1 x_2^0 x_4^1 \vee x_1^1 x_2^1 x_4^0.$$

*(End of Example)*

**Example 3.2** *Consider the function in Example 2.2.*

1) *SOPs for characteristic functions are given in Example 2.2. Each product is labeled as follows:*

$$r(0,1)=X_1^0 X_2^2 X_3^1 X_4^2,$$
$$r(0,2)=X_1^2 X_2^1 X_3^1 X_4^0,$$
$$r(1,1)=X_1^0 X_2^1 X_3^1 X_4^2,$$
$$r(1,2)=X_1^2 X_2^1 X_3^2 X_4^1,$$
$$r(2,1)=X_1^0 X_2^1 X_3^2 X_4^1,$$
$$r(2,2)=X_1^2 X_2^2 X_3^2 X_4^2,$$
$$r(3,1)=X_1^0 X_2^0 X_3^1 X_4^2,$$
$$r(3,2)=X_1^1 X_2^1 X_3^1 X_4^2.$$

2) *For each pair of products, find the variables whose products are null:*

$$s(0,1,1,1)=x_2$$
$$s(0,1,1,2)=x_1 \vee x_2 \vee x_3 \vee x_4$$
$$s(0,1,2,1)=x_1 \vee x_4$$
$$s(0,1,2,2)=x_3 \vee x_4$$
$$s(0,2,1,1)=x_2 \vee x_3 \vee x_4$$
$$s(0,2,1,2)=x_1 \vee x_3$$
$$s(0,2,2,1)=x_1 \vee x_3 \vee x_4$$
$$s(0,2,2,2)=x_2 \vee x_3 \vee x_4$$
$$s(0,3,1,1)=x_2$$
$$s(0,3,1,2)=x_1 \vee x_2$$
$$s(0,3,2,1)=x_1 \vee x_2 \vee x_4$$
$$s(0,3,2,2)=x_1 \vee x_4$$
$$s(1,2,1,1)=x_3 \vee x_4$$
$$s(1,2,1,2)=x_1 \vee x_2 \vee x_3$$
$$s(1,2,2,1)=x_1$$
$$s(1,2,2,2)=x_2 \vee x_4$$
$$s(1,3,1,1)=x_2$$
$$s(1,3,1,2)=x_1$$
$$s(1,3,2,1)=x_1 \vee x_2 \vee x_3 \vee x_4$$
$$s(1,3,2,2)=x_1 \vee x_3 \vee x_4$$
$$s(2,3,1,1)=x_2 \vee x_3 \vee x_4$$
$$s(2,3,1,2)=x_1 \vee x_3 \vee x_4$$
$$s(2,3,2,1)=x_1 \vee x_2 \vee x_3$$
$$s(2,3,2,2)=x_1 \vee x_2 \vee x_3$$

3) *$R$ is obtained as the logical product of $s(i,j,k,\ell)$, and we have $R = x_1 x_2 (x_3 \vee x_4)$. Thus, the function can be represented by the SOPs with variable set $\{X_1, X_2, X_3\}$:*

$$\mathcal{F}_1=X_1^0 X_2^1 X_3^1 \vee X_1^2 X_2^1 X_3^2,$$
$$\mathcal{F}_2=X_1^0 X_2^1 X_3^2 \vee X_1^2 X_2^2 X_3^2, \text{ and}$$
$$\mathcal{F}_3=X_1^0 X_2^0 X_3^1 \vee X_1^1 X_2^1 X_3^1.$$

*Or, the SOPs with the variable set $\{X_1, X_2, X_4\}$:*

$$\mathcal{F}_1=X_1^0 X_2^1 X_4^2 \vee X_1^2 X_2^1 X_4^1,$$
$$\mathcal{F}_2=X_1^0 X_2^1 X_4^1 \vee X_1^2 X_2^2 X_4^2, \text{ and}$$
$$\mathcal{F}_3=X_1^0 X_2^0 X_4^2 \vee X_1^1 X_2^1 X_4^2.$$

*(End of Example)*

## IV Analysis of Random Multiple-Valued Functions

In this Section, we will estimate the number of redundant variables in random incompletely specified multiple-valued functions. To make the problem easy to analyze, we use the following:

**Assumption 4.1** *In a* **random multiple-valued function** *$f$, if the probability of appearing $i$ in $f$ is $\alpha$, then the probability of appearing $i$ in the sub-function $f(|x_j = k)$ is also $\alpha$, where $i \in \{0,1,\ldots,q-1\}$ and $k \in \{0,1,\ldots,p-1\}$.*

This assumption implies that $n$, the number of the input variables, is sufficiently large. Furthermore, for simplicity, we assume that $0 < \alpha \ll 1$. Thus, in the function table for $f$, most entries are don't cares, and only a fraction of the entries are specified.

**Theorem 4.1** *Let $f : P^n \rightarrow Q$, where $P = \{0,1,\ldots,p-1\}$ and $Q = \{0,1,\ldots,q-1\}$ be a random function such that the probability taking the value $i$ $(i = 0,1,\ldots,q-1)$ is $\alpha$. Then, the probability that the variable $X_i$ is redundant in $f$ is given by*

$$\delta_1(n,p,q,\alpha) = \gamma_1^M,$$

*where $\gamma_1 = q(\alpha+\beta)^p - (q-1)\beta^p$, $\beta = 1 - q\alpha$, and $M = p^{n-1}$.*

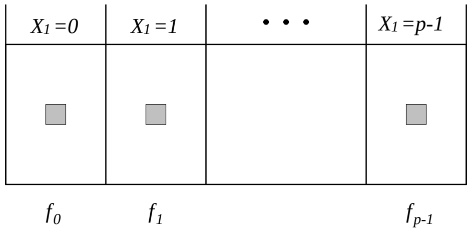


Figure 4.1: A map of $p$-valued function.

(Proof) Consider the map shown in Fig. 4.1, where $f$ is expanded with respect to $X_1$:

$$f = X_1^0 f_0 \vee X_1^1 f_1 \vee \cdots \vee X_1^{p-1} f_{p-1},$$

where $f_i = f(|X_1 = i)$. To be $X_1$ non-essential, $p$ sub-functions $f_i$ $(i = 0, 1, \ldots, p-1)$ must be compatible with each other. To be these $p$ sub-functions compatible, $p$ cells that correspond to $f(j, \boldsymbol{a})$ $(j = 0, 1, \ldots, p-1)$ must be compatible with each other, where $\boldsymbol{a}$ is a vector in $P^{n-1}$. In Fig. 4.1, these $p$ cells are hatched. These $p$ cells are compatible when the following conditions are satisfied:

- All the cells are don't cares. The probability of this case is given by $\beta^p$, where $\beta = 1 - q\alpha$.
- Only one cell takes value $i$ $(i = 0, 1, 2, \ldots, q-1)$ and others are don't cares (dc's). The probability of this case is given by $\binom{p}{1}\ \alpha^1 \beta^{p-1}$.
- Two cells take value $i$ and others are dc's. The probability of this case is given by $\binom{p}{2}\ \alpha^2 \beta^{p-2}$.

  $\cdots$
- All the cells take value $i$. The probability of this case is given by $\binom{p}{p}\ \alpha^p \beta^0$.

The probability $\gamma_1$ that these $p$ cells are compatible is derived as follows:

a) All the cells are dc's.

b) At least one cell is 0 and others dc's.

c) At least one cell is 1 and others dc's.

$\cdots$

d) At least one cell is $q-1$ and others dc's.

Since a) to d) are mutually disjoint, the probability $\gamma_1$ is given by

$$\begin{aligned}\gamma_1 &= q\left[\binom{p}{1}\alpha^1\beta^{p-1} + \binom{p}{2}\alpha^2\beta^{p-2} + \cdots + \binom{p}{p}\alpha^p\beta^0\right] + \beta^p \\ &= q[(\alpha+\beta)^p - \beta^p] + \beta^p.\end{aligned}$$

Note that $M = p^{n-1}$ cells exist in the sub-function $f_0$. To be $X_1$ non-essential, the same thing must hold for each of these $M$ cells. Thus, the probability that $X_1$ is non-essential in $f$ is given by $\delta_1(n, p, q, \alpha) = \gamma_1^M$. □

**Corollary 4.1**

*1) Let $p = 2$ and $q = 2$. We have $\gamma_1 = 2(\alpha+\beta)^2 - \beta^2$, and $\beta = 1 - 2\alpha$. Thus,*

$$\gamma_1 = 2(1-\alpha)^2 - (1-2\alpha)^2 = 1 - 2\alpha^2,$$
$$\delta_1(n, 2, 2, \alpha) = \gamma_1^{2^{n-1}}.$$

*2) Let $p = 3$ and $q = 3$. We have $\gamma_1 = 3(\alpha+\beta)^3 - 2\beta^3$, and $\beta = 1 - 3\alpha$. Thus,*

$$\gamma_1 = 3(1-2\alpha)^3 - 2(1-3\alpha)^3 = 1 - 18\alpha^2 + 30\alpha^3,$$
$$\delta_1(n, 3, 3, \alpha) = \gamma_1^{3^{n-1}}.$$

*3) Let $p = 2$ and $q = 4$. We have $\gamma_1 = 4(\alpha+\beta)^2 - 3\beta^2$, and $\beta = 1 - 4\alpha$. Thus,*

$$\gamma_1 = 4(1-3\alpha)^2 - 3(1-4\alpha)^2 = 1 - 12\alpha^2,$$
$$\delta_1(n, 2, 4, \alpha) = \gamma_1^{2^{n-1}}.$$

*4) Let $p = 4$ and $q = 2$. We have $\gamma_1 = 2(\alpha+\beta)^4 - \beta^4$, and $\beta = 1 - 2\alpha$. Thus,*

$$\gamma_1 = 2(1-\alpha)^4 - (1-2\alpha)^4 = 1 - 12\alpha^2 + 24\alpha^3 - 14\alpha^4,$$
$$\delta_1(n, 4, 2, \alpha) = \gamma_1^{4^{n-1}}.$$

*5) Let $p = 4$ and $q = 4$. We have $\gamma_1 = 4(\alpha+\beta)^4 - 3\beta^4$, and $\beta = 1 - 4\alpha$. Thus,*

$$\gamma_1 = 4(1-3\alpha)^4 - 3(1-4\alpha)^4 = 1 - 72\alpha^2 + 336\alpha^3 - 444\alpha^4,$$
$$\delta_1(n, 4, 4, \alpha) = \gamma_1^{4^{n-1}}.$$

**Theorem 4.2** *Let $f : P^n \to Q$, where $P = \{0, 1, \ldots, p-1\}$ and $Q = \{0, 1, \ldots, q-1\}$ be a random function such that the probability of taking the value $i$ $(i = 0, 1, \ldots, q-1)$ is $\alpha$. Then, the probability that $f$ is a redundant with $\{X_1, X_2, \ldots, X_k\}$ is $\delta_k = \gamma_k^M$, where $\gamma_k = q(\alpha+\beta)^{p^k} - (q-1)\beta^{p^k}$, $\beta = 1 - q\alpha$, and $M = p^{n-k}$.*

(Proof) Assume that $f$ is expanded with respect to $X_1$, $X_2$, … , and $X_k$:

$$\begin{aligned}f = &X_1^0 X_1^0 \cdots X_k^0 f_{00\cdots 0} \vee X_1^0 X_1^0 \cdots X_k^1 f_{00\cdots 1} \vee \cdots \\ &\vee X_1^{p-1} X_1^{p-1} \cdots X_k^{p-1} f_{p-1\, p-1\, \cdots\, p-1}\end{aligned}$$

$\{X_1, X_2, \ldots, X_k\}$ is a redundant set in $f$ if $p^k$ sub-functions $f_{00\cdots 0}$, … , $f_{p-1\, p-1\, \cdots\, p-1}$ are mutually compatible. To be these $p^k$ sub-functions compatible, $p^k$ cells that correspond to $f(\boldsymbol{a}, 0, 0, \ldots, 0)$ must be compatible, where $\boldsymbol{a}$ is a vector in $P^k$. Similar to the proof of Theorem 4.1, we have

$$\begin{aligned}\gamma_k &= q\left[\binom{p^k}{1}\alpha^1\beta^{p^k-1} + \binom{p^k}{2}\alpha^2\beta^{p^k-2} + \cdots + \binom{p^k}{p^k}\alpha^{p^k}\beta^0\right] + \beta^{p^k} \\ &= q[(\alpha+\beta)^{p^k} - \beta^{p^k}] + \beta^{p^k}.\end{aligned}$$

Note that $M = p^{n-k}$ cells exist in the sub-function $f_{00\cdots0}$. When $f$ is redundant in $\{X_1, X_2, \ldots, X_k\}$, the same thing hold for each of these $M$ cells. Thus, the probability that $\{X_1, X_2, \ldots, X_k\}$ is a redundant set is $\delta_k = \gamma_k^M$. □

**Theorem 4.3** *In an $n$ variable function $f$, let $\delta_k$ be the probability that the set of $k$ variables is redundant. Then, we have the following:*

1) *The probability that at least one variable is redundant in $f$ is*

$$\theta_1 = 1 - (1 - \delta_1)^n.$$

2) *The probability that $f$ has at least one set of redundant variables with size two is*

$$\theta_2 = 1 - (1 - \delta_2)^{n(n-1)/2}.$$

3) *The probability that $f$ has at least one set of redundant variables with size three is*

$$\theta_3 = 1 - (1 - \delta_3)^{n(n-1)(n-2)/6}.$$

(Proof) The probability that $X_i$ is essential is $1 - \delta_1$. The probability that all the variables are essential is $(1 - \delta_1)^n$. Thus, we have $\theta_1 = 1 - (1 - \delta_1)^n$. Similarly, we have $\theta_2$ and $\theta_3$. □

**Definition 4.1** *A property A is said to hold for almost all functions if the proportion of $n$-variable functions that do not satisfy A tends to zero as $n \to \infty$. In other words, let $w(n)$ be the number of $n$-variable functions that do not satisfy A. Then $w(n)/2^{2^n} \to 0$ as $n \to \infty$.*

**Theorem 4.4** *Let $p = q = 2$. If $\alpha \leq \lambda 2^{-(n-1)/2}$, then at least one variable is redundant for almost all functions, where $\lambda$ is a positive constant.*

(Proof) When $\alpha = \lambda 2^{-(n-1)/2}$, we have $\alpha^2 = \frac{\lambda^2}{M}$, where $M = 2^{n-1}$. Thus, $\gamma_1 = 1 - 2\alpha^2 = 1 - \frac{2\lambda^2}{M}$. Note that for sufficiently large $n$, $0 < \frac{2\lambda^2}{M} << 1$ and $M$ is very large. In this case, we have

$$\delta_1 = \gamma_1^M = (1 - \frac{2\lambda^2}{M})^M \simeq e^{-2\lambda^2} \simeq (0.135)^{\lambda^2}.$$

By Theorem 4.3, the probability that at least one variable is redundant is given by $\theta_1 = 1 - (1 - \delta_1)^n$. As $n \to \infty$, $(1 - \delta_1)^n \to 0$. Thus, for almost all functions at least one variable is redundant. □

**Theorem 4.5** *Let $p = q = 3$. If $\alpha \leq \lambda 3^{-(n+1)/2}$, then at least one variable is redundant for almost all functions, where $\lambda$ is a positive constant.*

(Proof) When $\alpha = \lambda 3^{-(n+1)/2}$, we have $\alpha^2 = \frac{\lambda^2}{9M}$, where $M = 3^{n-1}$. Note that $\gamma_1 = 1 - 18\alpha^2 + 30\alpha^3$. Since $0 < \alpha \ll 1$, we have

$$\gamma_1 \simeq 1 - \frac{2\lambda^2}{M}.$$

Similar to the proof of Theorem 4.4, we can show that at least one variable is redundant for almost all functions. □

**Theorem 4.6** *Let $p = q = 4$. If $\alpha \leq \lambda 2^{-n}/3$, then at least one variable is redundant for almost all functions, where $\lambda$ is a positive constant.*

(Proof) When $\alpha = \lambda 2^{-n}/3$, we have $\alpha^2 = \frac{\lambda^2}{36M}$, where $M = 4^{n-1}$. Note that $\gamma_1 = 1 - 72\alpha^2 + 336\alpha^3 - 444\alpha^4$. Since $0 < \alpha \ll 1$, we have

$$\gamma_1 \simeq 1 - \frac{2\lambda^2}{M}.$$

Similar to the proof of Theorem 4.4, we can show that at least one variable is redundant in $f$. □

# V Experimental Results

We developed a program to find maximal redundant sets of input variables for incompletely specified multiple-valued functions. Table 5.1 summarizes the results of statistical analysis and computer simulation.

For each set of parameters $(p, q, n, N_{min})$, we randomly generated 1000 sample functions and found maximal numbers of redundant variables. $N_{min}$ denotes the number of combinations that are mapped to $i$ $(i = 0, 1, \ldots, q-1)$. That is, for the case of $q = 2$, $f$ has $N_{min}$ true minterms and $N_{min}$ false minterms. The number of unspecified minterms is $p^n - qN_{min}$.

## 5.1 Experiment Supporting Theorem 4.3

In Table 5.1, the column headed by $\theta_1$ shows the probability that $f$ has at least one redundant variable, and the column headed by $\theta_2$ shows the probability that $f$ has at least one set of redundant variables with size two. The values of $\theta_1$ and $\theta_2$ were calculated by using the formulas in Theorem 4.3.

The column denoted by $k$ $(k = 0, 1, 2, 3)$ shows the number of functions that have the sets of redundant variables with maximum size $k$. These values were obtained by computer simulation.

For example, when $p = 2$, $q = 2$, $n = 17$ and $N_{min} = 512$, the statistical analysis shows that $\theta_1 = 0.91588$ and $\theta_2 = 0.29779$. On the other hand, the computer simulation shows that out of 1000 functions, 86 functions have no redundant variables (i.e., all the variables are essential); for 683 functions, the sizes of the maximal set of redundant variables are one; and for 231

Table 5.1: Comparison of statistical analysis and computer simulation.

| | | | | Statistical Analysis | | Computer Simulation | | | |
|---|---|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $n$ | $N_{min}$ | $\theta_1$ | $\theta_2$ | 0 | 1 | 2 | 3 |
| 2 | 2 | 9 | 32 | .72684 | .15188 | 310 | 631 | 59 | 0 |
| 2 | 2 | 11 | 64 | .79733 | .17411 | 221 | 664 | 105 | 0 |
| 2 | 2 | 13 | 128 | .84883 | .20645 | 162 | 688 | 150 | 0 |
| 2 | 2 | 15 | 256 | .88706 | .24837 | 113 | 701 | 185 | 0 |
| 2 | 2 | 16 | 362 | .90246 | .27258 | 101 | 664 | 235 | 0 |
| 2 | 2 | 17 | 512 | .91558 | .29779 | 86 | 683 | 231 | 0 |
| 2 | 2 | 18 | 724 | .92710 | .32491 | 72 | 665 | 263 | 0 |
| 2 | 4 | 8 | 10 | .54565 | .06111 | 573 | 424 | 3 | 0 |
| 2 | 4 | 10 | 20 | .63330 | .06620 | 417 | 558 | 25 | 0 |
| 2 | 4 | 12 | 40 | .70149 | .07534 | 328 | 627 | 45 | 0 |
| 2 | 4 | 14 | 100 | .30503 | .00207 | 723 | 274 | 3 | 0 |
| 4 | 2 | 6 | 50 | .66652 | .00585 | 369 | 630 | 1 | 0 |
| 4 | 2 | 8 | 120 | .99711 | .68302 | 1 | 430 | 569 | 0 |
| 4 | 2 | 9 | 240 | .99859 | .75767 | 1 | 367 | 632 | 0 |
| 3 | 3 | 5 | 9 | .55571 | .03105 | 563 | 437 | 0 | 0 |
| 3 | 3 | 7 | 27 | .65391 | .01849 | 336 | 634 | 0 | 0 |
| 3 | 3 | 9 | 81 | .73493 | .01720 | 281 | 709 | 0 | 0 |
| 3 | 3 | 11 | 243 | .79959 | .02707 | 207 | 772 | 21 | 0 |
| 4 | 4 | 4 | 5 | .58522 | .02261 | 563 | 437 | 0 | 0 |
| 4 | 4 | 6 | 21 | .62422 | .00359 | 403 | 597 | 0 | 0 |
| 4 | 4 | 8 | 85 | .69828 | .00205 | 309 | 691 | 0 | 0 |
| 4 | 4 | 9 | 170 | .73082 | .00217 | 258 | 740 | 2 | 0 |

functions, the sizes of the maximal set of redundant variables are two. The formula of Theorem 4.3 shows that for 91.6% of the functions have at least one redundant variables, while computer simulation shows that $683 + 231 = 914$ functions out of 1000 have redundant set with size one or two. In this case, the formula predicts the sizes of the redundant variable set quite well.

### 5.2 Experiment Supporting Theorem 4.4

In Table 5.1, for $p = q = 2$, values for $N_{min}$ were selected to satisfy $\alpha = \frac{N_{min}}{2^{n-1}} = \lambda 2^{-(n-1)/2}$, where $\lambda = 2$. Thus, $N_{min} = 2^{(n+1)/2}$. Table 5.1 shows that as $n$ increases, the value of $\theta_1$ increases monotonously to approach 1.00. Similar tendency can be observed in the cases of $p = q = 3$ and $p = q = 4$.

## VI Conclusion and Comments

In this paper, we have

1) Formulated the minimization problem of dependent variables for incompletely specified multiple-valued logic functions, and shown the algorithm.
2) Derived the probability that a given set of variable is redundant in a randomly generated function.
3) Shown conditions that at least one variable is redundant in randomly generated functions.
4) Verified the usefulness of the approach by computer simulation.

In this paper, we used statistical analysis. It is possible to use combinatorial analysis instead [10]. In such a case, the estimation is accurate even when $n$ is small.

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