Totally Undecomposable Functions: Applications to Efficient Multiple-Valued Decompositions

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Abstract

A function $f : P^n \to P$, $P = \{0, 1, \ldots, p-1\}$ is k-decomposable iff f can be represented as $f(X_1, X_2) = g(h_1(X_1), h_2(X_1), \ldots, h_k(X_1), X_2)$, where (X_1, X_2) is a bipartition of input variables. This paper introduces the notion of totally k-undecomposable functions. By using this concept, we can drastically reduce the search space to find k-decompositions. A systematic method to find the bipartitions of input variables that will not produce any k-decompositions is presented. By combining it to the conventional decomposition methods, we can build an efficient functional decomposition system. This method is promising to design LUT-based FPGAs.

Key words: Functional decomposition, Symmetric function, LUT-based FPGA, Multiple-valued logic function.

I Introduction

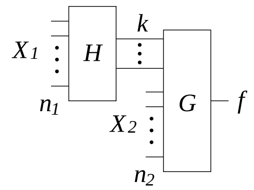
Decompositions of logic functions have been studied for many years. Major contributions are summarized as follows:

- Formulations using decomposition tables [1, 5].
- Formulations using compatibility [12].
- Fast method using Jacobian [20].
- Applications to multi-level PLA networks [13, 6].
- Extension to incompletely specified functions [21].
- Computation of column multiplicity using BDDs [14, 8, 4].
- Application to FPGAs [11].
- Bi-decomposition [15].
- Fast method [2, 10].
- Extension to multiple-valued logic [7, 9].
- Extension to multiple-output functions [18, 22, 9].

In the above contributions, most are related to two-valued functions. However, extensions to multiple-valued functions are quite natural.

In this paper, we will consider decompositions shown in Fig. 1.1. Given a multiple-valued function $f : P^n \to P$, $P = \{0, 1, \ldots, p-1\}$, we will consider the problem whether f can be represented as $f(X_1, X_2) = g(h_1(X_1), h_2(X_1), \ldots, h_k(X_1), X_2)$, or not.

Let n be the number of the input variables, then we have to consider nearly 2^n different bipartitions (X_1, X_2) of the input variables $\{x_1, x_2, \ldots, x_n\}$. When



$$f(X_1, X_2) = g(h_1(X_1), h_2(X_1), \ldots, h_k(X_1), X_2)$$

Figure 1.1: Disjoint k-decomposition.

n is large, the number of bipartitions to consider is too large, and the exhaustive search is impractical.

This paper introduces the concept of totally undecomposable functions. By using this concept, we can drastically reduce computation time to find decompositions. This paper shows a systematic method to find the bipartitions of input variables that will not produce any decompositions. By combining it to the conventional decomposition methods, we can build an efficient functional decomposition system.

The rest of this paper is organized as follows: Section II gives definitions and basic properties of functional decompositions. Section III introduces the concept of k-undecomposable functions. It also derives a theorem to find bipartitions (X_1, X_2) that will not produce any k-decomposition. Section IV shows a method to represent a set of bipartitions by using a switching function. Section V enumerates the number of k-undecomposable functions. It also shows that, for sufficiently large n, almost all functions are totally k-undecomposable.

II Definitions and Basic Properties

Definition 2.1 *A p-**valued function** is a mapping $f : P^n \to P$, where $P = \{0, 1, \ldots, p-1\}$ and $p \geq 2$. If $p = 2$, f is a **switching function**.*

Definition 2.2 *Let the set of the input variables be $\{X\} = \{x_1, x_2, \ldots, x_n\}$. $(X_1, X_2, \ldots, X_r)$ is a **partition** of X if $\{X_i\} \cap \{X_j\} = \phi$ $(1 \leq i < j \leq n)$ and $\{X_1\} \cup \{X_2\} \cup \cdots \cup \{X_r\} = \{X\}$. Especially when $r = 2$, the partition is a **bipartition**. The number of the variables in $\{X\}$ is denoted by $|X|$.*

Definition 2.3 *A p-valued function f has a **disjoint k-decomposition** iff f is represented as $f(X_1, X_2) =$*

$X_2 = (x_3, x_4)$ \ $X_1 = (x_1, x_2)$	00	01	02	10	11	12	20	21	22
00	2	1	1	2	2	2	1	1	1
01	0	1	1	0	0	0	1	1	1
02	1	1	1	1	1	1	1	0	0
10	0	0	0	0	0	0	0	0	0
11	1	0	0	1	1	1	0	2	2
12	2	0	0	2	2	2	0	2	2
20	0	2	2	0	0	0	2	2	2
21	1	2	2	1	1	1	2	0	0
22	2	2	2	2	2	2	2	0	0

Figure 2.1: Decomposition table.

$g(h_1(X_1), h_2(X_1), \ldots, h_k(X_1), X_2)$, *where* (X_1, X_2) *is a bipartition of* X, *and* g *and* h_i *are* p*-valued functions. If* $|X_1| \geq k+1$ *and* $|X_2| \geq 1+\lceil \log_p k \rceil$, *then the decomposition is* **non-trivial**, *and* f *is* k**-decomposable**, *where* $\lceil a \rceil$ *denotes the least integer not smaller than* a. *We also assume that functions with up to two variables are decomposable.* $\{X_1\}$ *and* $\{X_2\}$ *are the* **bound set** *and the* **free set**, *respectively. Variables in* $\{X_1\}$ *and* $\{X_2\}$ *are* **bound variables** *and* **free variables**, *respectively. When* f *is* k*-decomposable,* f *is realized by the network shown in Fig. 1.1.*

Definition 2.4 *If* f *does not depend on one or more variables, then* f *is* **degenerate**.

Note that if f is degenerate, then f is decomposable.

Definition 2.5 *Let* $f(X)$ *be a* p*-valued function, and* (X_1, X_2) *be a bipartition of* X, *where* $n_1 = |X_1|$ *and* $n_2 = |X_2|$. *The* **decomposition table** *of* f *has* p^{n_1} *columns and* p^{n_2} *rows, each column has distinct* p*-ary label of* n_1 *digits, each row has distinct* p*-ary label of* n_2 *digits, and the corresponding entry of the table represents the value of* f.

Example 2.1 *Let* $f(X)$ *be a function* $f : \{0,1,2\}^4 \rightarrow \{0,1,2\}$, *and* (X_1, X_2) *be a bipartition of* X, *where* $X_1 = (x_1, x_2)$ *and* $X_2 = (x_3, x_4)$. *Fig. 2.1 is an example of a decomposition table.* ∎

Definition 2.6 *The number of different column patterns in the decomposition table for a bipartition* (X_1, X_2) *is the* **column multiplicity** *and is denoted by* $\mu(f : X_1, X_2)$.

Theorem 2.1 *A* p*-valued function* $f(X)$ *has a disjoint* k*-decomposition* $f(X) = g(h_1(X_1), h_2(X_1), \ldots, h_k(X_1), X_2)$ *iff* $\mu(f : X_1, X_2) \leq p^k$.

The size of decomposition tables for n variables is p^n, and the number of different bipartitions is $O(2^n)$. Thus, the straightforward method to find a k-decomposition is impractical for the functions with many inputs. A method to find decompositions by using ROBDDs (reduced ordered binary decision diagrams) or ROMDDs (reduced ordered multi-valued decision diagrams) has been developed.

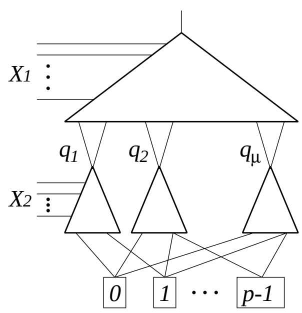


Figure 2.2: Computation of column multiplicity $\mu(f : X_1, X_2)$.

Theorem 2.2 *[14, 8, 7] Let* (X_1, X_2) *be a bipartition of* X. *Suppose that the ROMDD for* $f(X)$ *is partitioned into two blocks as shown in Fig. 2.2. The number of nodes in the lower block that are adjacent to the upper block is equal to* $\mu(f : X_1, X_2)$.

Lemma 2.1 *For any bipartition* (X_1, X_2) *of input variables of a* p*-valued function* f, $1 \leq \mu(f : X_1, X_2) \leq \min(p^{n_1}, p^{p^{n_2}})$, *where* $n_1 = |X_1|$ *and* $n_2 = |X_2|$.

(Proof) The number of columns in the decomposition table is p^{n_1}. Thus, we have $\mu(f : X_1, X_2) \leq p^{n_1}$. The number of different functions of n_2 variables is $p^{p^{n_2}}$. Since each column of the decomposition table shows an n_2-variable function, we have $\mu(f : X_1, X_2) \leq p^{p^{n_2}}$. □

Definition 2.7 *Let* $f(X_A, X_B)$ *be a function, where* $|X_B| = n_B$. *Let* $\vec{a}_B \in P^{n_B}$ *be an assignment for* X_B. *Then,* $f(X_A, \vec{a}_B)$ *denotes the sub-function, where the values of* X_B *are fixed to the constants* $\vec{a}_B$. $f(\vec{a}_A, X_B)$ *is similarly defined.*

Definition 2.8 *Let* $X = (x_1, x_2, \ldots, x_n)$ *and* $\vec{a} = (a_1, a_2, \ldots, a_n)$. *Then,*

$$X^{\vec{a}} \begin{cases} = p-1 \text{ if } x_i = a_i \text{ for } i = 1, 2, \ldots, n. \\ = 0 \text{ otherwise.} \end{cases}$$

Lemma 2.2 *If* $k \geq p^{|X_2|}$, *then any* p*-valued function is realized in the network shown in Fig. 1.1.*

(Proof) Let $n_2 = |X_2|$. An arbitrary function $f(X_1, X_2)$ is represented by $f(X_1, X_2) = \bigvee_{\vec{a} \in P^{n_2}} f(X_1, \vec{a}) X_2^{\vec{a}}$, where $P = \{0, 1, \ldots, p-1\}$. Since the number of products in the above expression is at most p^{n_2}, we have the lemma. □

Example 2.2 *When* $n_2 = 1$ *and* $k = p$, *any* p*-valued function is realized in the network shown in Fig. 2.3 by using the following expansion:*

$$f(X_1, X_2) = X_2^0 f_0(X_1) \vee X_2^1 f_1(X_1) \vee \cdots \vee X_2^{p-1} f_{p-1}(X_1).$$

∎

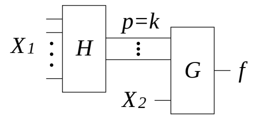


Figure 2.3: Example of a trivial k-decomposition.

	$X_1 = (x_1, x_2, x_3)$							
	0	0	0	0	1	1	1	1
	0	0	1	1	0	0	1	1
$X_2 = (x_4, x_5)$	0	1	0	1	0	1	0	1
00	1	1	0	0	1	0	0	0
01	1	0	0	1	0	1	1	0
10	0	0	0	1	0	1	1	0
11	0	1	1	0	0	0	0	0

Figure 3.1: Totally 2-undecomposable function.

We assume that the k-decomposition in Lemma 2.2 is trivial. This is why we assumed that $n_2 \geq \lceil \log_p k \rceil + 1$ in Definition 2.3.

III k-Undecomposable Functions

In this part, we introduce the notion of totally k-undecomposable functions. We will show that if $f(X_A, \vec{a}_B)$ is totally k-undecomposable, then $f(X_A, X_B)$ is k-undecomposable for many bipartitions. Thus, by finding totally k-undecomposable subfunctions, we can drastically reduce the search space for functional decompositions.

Definition 3.1 *A p-valued function $f(X)$ is* **totally k-undecomposable** *if $\mu(f : X_1, X_2) > p^k$ for any bipartition (X_1, X_2), where $|X_1| \geq 1 + k$ and $|X_2| \geq 1 + \lceil \log_p k \rceil$.*

Example 3.1 *Consider the case where $n = 3$, $p = 2$, and $k = 1$. $f(x_1, x_2, x_3) = x_1x_2 \vee x_2x_3 \vee x_3x_1$ is totally 1-undecomposable.* ∎

Example 3.2 *Consider the case where $n = 5$, $p = 2$, and $k = 2$. A five-variable function f shown in Fig. 3.1 is totally 2-undecomposable, since $\mu(f : X_1, X_2) > 4$ for any bipartitions with $|X_1| = 3$ and $|X_2| = 2$.* ∎

Lemma 3.1 *Let $(X_{1A}, X_{1B}, X_{2A}, X_{2B})$ be a partition of X, where $|X_{1A}| \geq k + 1$ and $|X_{2A}| \geq 1 + \lceil \log_p k \rceil$. Let $\vec{a}_{1B}$ and $\vec{a}_{2B}$ be assignments of X_{1B} and X_{2B}, respectively. If $f(X_{1A}, \vec{a}_{1B}, X_{2A}, \vec{a}_{2B})$ has no k-decomposition of the form*

$$\hat{f}(X_{1A}, X_{2A}) = f(X_{1A}, \vec{a}_{1B}, X_{2A}, \vec{a}_{2B}) = \hat{g}(\hat{h}_1(X_{1A}), \hat{h}_2(X_{1A}), \ldots, \hat{h}_k(X_{1A}), X_{2A}),$$

then, $f(X_{1A}, X_{1B}, X_{2A}, X_{2B})$ has no k-decomposition of the form

$$f(X_{1A}, X_{1B}, X_{2A}, X_{2B}) = g(h_1(X_{1A}, X_{1B}), h_2(X_{1A}, X_{1B}), \ldots, h_k(X_{1A}, X_{1B}), X_{2A}, X_{2B}).$$

In this case, $\{X_{1B}\}$ or $\{X_{2B}\}$ can be empty set(s).

(Proof) Assume that f has a k-decomposition of the form

$$f(X_{1A}, X_{1B}, X_{2A}, X_{2B}) = g(h_1(X_{1A}, X_{1B}), h_2(X_{1A}, X_{1B}), \ldots, h_k(X_{1A}, X_{1B}), X_{2A}, X_{2B}).$$

Assign $\vec{a}_{1B}$ and $\vec{a}_{2B}$ to X_{1B} and X_{2B}, respectively. Then, we have the decomposition $f(X_{1A}, \vec{a}_{1B}, X_{2A}, \vec{a}_{2B}) = g(h_1(X_{1A}, \vec{a}_{1B}), h_2(X_{1A}, \vec{a}_{1B}), \ldots, h_k(X_{1A}, \vec{a}_{1B}), X_{2A}, \vec{a}_{2B})$. However, this contradicts the assumption of the lemma. □

Example 3.3 *Consider the case where $n = 8$, $k = 2$, and $p = 2$. Let $f(x_1, x_2, \ldots, x_8)$ be an 8-variable function. If $(x_1, x_2, x_3, 0, 1, x_6, x_7, 1)$ has no 2-decomposition of the form $\hat{f}(x_1, x_2, x_3, x_6, x_7) = \hat{g}(\hat{h}_1(x_1, x_2, x_3), \hat{h}_2(x_1, x_2, x_3), x_6, x_7)$, then $f(x_1, x_2, \ldots, x_8)$ has no 2-decomposition of the form $f = g(h_1(x_1, x_2, x_3, x_4, x_5), h_2(x_1, x_2, x_3, x_4, x_5), x_6, x_7, x_8)$. In this example, $X_{1A} = (x_1, x_2, x_3)$, $X_{1B} = (x_4, x_5)$, $X_{2A} = (x_6, x_7)$, $X_{2B} = (x_8)$, $\vec{a}_{1B} = (0, 1)$, and $\vec{a}_{2B} = 0$.* ∎

Theorem 3.1 *Let (X_A, X_B) be a partition of X, where $|X_A| \geq k + \lceil \log_p k \rceil + 2$ and $|X_B| \geq 1$. For an assignment $\vec{a}_B$, if $f(X_A, \vec{a}_B)$ is totally k-undecomposable, then f has no decomposition of the form $f(X_1, X_2) = g(h_1(X_1), h_2(X_1), \ldots, h_k(X_1), X_2)$, where (X_1, X_2) is a bipartition of X, $|\{X_A\} \cap \{X_1\}| \geq k + 1$, and $|\{X_A\} \cap \{X_2\}| \geq 1 + \lceil \log_p k \rceil$.*

(Proof) Let $X_A = (X_{1A}, X_{2A})$ and $X_B = (X_{1B}, X_{2B})$. Then, apply Lemma 3.1, and we have the theorem. □

Definition 3.2 *Let (X_1, X_2) be a bipartition of $\{x_1, x_2, \ldots, x_n\}$, where $X_1 = (x_1, x_2, \ldots, x_r)$ and $X_2 = (x_{r+1}, x_{r+2}, \ldots, x_n)$. Such a bipartition is compactly denoted by the bipartition of integers $(1, 2, \ldots, r|r + 1, r + 2, \ldots, n)$.*

Example 3.4 *Let $f(x_1, x_2, x_3, x_4, x_5)$ be a five-variable two-valued function. If $f(x_1, x_2, x_3, 0, 0)$ is totally 1-undecomposable, then f is 1-undecomposable for the following 12 bipartitions:*

$$(1, 2|3, 4, 5), (1, 2, 4|3, 5), (1, 2, 5|3, 4), (1, 2, 4, 5|3),$$
$$(1, 3|2, 4, 5), (1, 3, 4|2, 5), (1, 3, 5|2, 4), (1, 3, 4, 5|2),$$
$$(2, 3|1, 4, 5), (2, 3, 4|1, 5), (2, 3, 5|1, 4), (2, 3, 4, 5|1).$$

∎

Theorem 3.2 *Consider a p-valued function $f(X_A, X_B)$, where $n_A = |X_A| \geq k + \lceil \log_p k \rceil + 2$ and $n_B = |X_B| \geq 1$. For an assignment $\vec{a}_B$, if $f(X_A, \vec{a}_B)$ is totally k-undecomposable, then f is k-undecomposable for*

$$\alpha(n_A, n_B, p, k) = \left[\sum_{i=k+1}^{n_A-1-\lceil \log_p k \rceil} C(n_A, i) \right] 2^{n_B}$$

bipartitions.

(Proof) F is k-undecomposable when the following conditions are satisfied:

1) More than k variables in $\{X_A\}$ are included as bound variables.
2) More than $\lceil \log_p k \rceil$ variables in $\{X_A\}$ are included as free variables.
3) Variables in $\{X_B\}$ can be either in the bound set or the free set.

From 1) and 2), we have the first factor. And, from 3), we have the second factor. □

Example 3.5 *Let $f(x_1, x_2, x_3, x_4, x_5)$ be a five-variable 2-valued function. If $f(x_1, x_2, x_3, 0, 0)$ is totally 1-undecomposable, then f is 1-undecomposable for $\alpha = 12$ bipartitions, since $k = 1$, $p = 2$, $n_1 = 3$, and $n_2 = 2$. This is also verified by Example 3.4* ∎

Corollary 3.1 *Consider an n-variable function $f(X_A, X_B)$, where $n_A = |X_A| \geq k + \lceil \log_p k \rceil + 2$, and $n_B = |X_B| \geq 1$. For an assignment $\vec{a}_B$, if $(X_A, \vec{a}_B)$ is totally k-undecomposable, then we have to check for at most*

$$\beta(n_A, n_B, p, k) = \left[\sum_{i=0}^{k} C(n_A, i) + \sum_{j=0}^{\lceil \log_p k \rceil} C(n_A, j) \right] 2^{n_B}$$

bipartitions.

(Proof) There are $2^n = 2^{n_A} 2^{n_B}$ bipartitions. Among them, $\alpha(n_A, n_B, p, k)$ bipartitions are k-undecomposable. So, we have to check at most $\beta(n_A, n_B, p, k) = 2^n - \alpha(n_A, n_B, p, k)$ bipartitions. □

Example 3.6 *Corollary 3.1 shows that when $p = 2$ and $k = 1$, the fraction of β to 2^n is $\gamma = \frac{\beta}{2^n} = \frac{n_A+2}{2^{n_A}}$. Therefore, when $n_A = 3$, $\gamma = 5/8$; when $n_A = 4$, $\gamma = 3/8$; when $n_A = 5$, $\gamma = 7/32$; and when $n_A = 6$, $\gamma = 1/8$.* ∎

IV Switching Function Representing Set of Bipartitions

Functional decomposition is to find a bipartition (X_1, X_2) such that $f(X_1, X_2) = g(h_1(X_1), h_2(X_1), \ldots, h_k(X_1), X_2)$. There are 2^n different bipartitions including trivial ones, and these can be represented by a switching function of n variables. In this part, we will introduce such representations. Also, bipartitions that will not produce decompositions are compactly denoted by symmetric functions. We also introduce notations for symmetric functions.

Definition 4.1 *A function f is a* **totally symmetric function** *if any permutation of the variables in f does not change the function.*

Definition 4.2 *The* **elementary symmetric functions** *of n variables are*

$S_0^n = \bar{x}_1 \bar{x}_2 \cdots \bar{x}_n$,
$S_1^n = x_1 \bar{x}_2 \cdots \bar{x}_n \vee \bar{x}_1 x_2 \bar{x}_3 \cdots \bar{x}_n \vee \cdots \vee \bar{x}_1 \bar{x}_2 \cdots \bar{x}_{n-1} x_n$,
$\cdots\cdots\cdots\cdots\cdots\cdots$, *and*
$S_n^n = x_1 x_2 \cdots x_n$.

$S_i^n = 1$ iff exactly i inputs are equal to one. Let $A \subseteq \{0, 1, \ldots, n\}$. A symmetric function S_A^n is defined as follows:

$$S_A^n = \bigvee_{i \in A} S_i^n.$$

Example 4.1 *$f(x_1, x_2, x_3) = x_1 x_2 x_3 \vee x_1 \bar{x}_2 \bar{x}_3 \vee \bar{x}_1 x_2 \bar{x}_3 \vee \bar{x}_1 \bar{x}_2 x_3$ is a totally symmetric function. $f = 1$ when all the variables are one, or when only one variable is one. Thus, f can be written as $S_1^3 \vee S_3^3 = S_{\{1,3\}}^3$.* ∎

Definition 4.3 *A set of bipartitions of the input variables $\{x_1, x_2, \ldots, x_n\}$ is represented by a switching function bp of n variables. In bp, $x_i = 1$ denotes that x_i is in the bound set, and $x_i = 0$ denotes that x_i is in the free set. The number of true minterms of bp is denoted by $|bp|$.*

Example 4.2 *Suppose that $n = 5$. The minterm $x_1 x_2 x_3 \bar{x}_4 \bar{x}_5$ denotes that x_1, x_2, and x_3 are in the bound set, and x_4 and x_5 are in the free set.* ∎

Lemma 4.1 *The set of bipartitions for trivial k-decompositions for n-variable p-valued function is given by*

$$u_0 = S_{\{0,1,\ldots,k\}}^n \vee S_{\{n-\lceil \log_p k \rceil, \ldots, n\}}^n.$$

(Proof) When the number of variables in the bound set is less than $k+1$, then it is a trivial decomposition. To be non-trivial k-decomposition, at least $1 + \lceil \log_p k \rceil$ variables must be in the free set. So, if the number of variables in the bound set is greater than $n - 1 - \lceil \log_p k \rceil$, then it is a trivial decomposition. □

Example 4.3 *Let $n = 10$, $p = 2$, and $k = 2$, then the set of bipartitions for trivial k-decompositions is given by $u_0 = S_{\{0,1,2,9,10\}}^{10}$. This is explained as follows: If the number of bound variables is two or smaller, then* *the decomposition is trivial, since the module for H has two outputs. If the number of variables in the bound set is 9 or 10, then the number of free variables is one or zero. By Definition 2.3, this also corresponds to a trivial decomposition. Thus, the number of trivial decompositions is given by*

$$|u_0| = C(10,0) + C(10,1) + C(10,2) \\ +C(10,9) + C(10,10) = 67.$$

The set of non-trivial bipartitions is given by $\bar{u}_0 = S^{10}_{\{3,4,5,6,7,8\}}$. ∎

Theorem 4.1 *Let $f(X)$ be a p-valued function, and (X_A, X_B) be a partition of X. If $f(X_A, \vec{a}_B) = \hat{f}(x_1, x_2, \ldots, x_r)$ is totally k-undecomposable, then f has no k-decomposition for the bipartitions*

$$u = S^r_{\{k+1,k+2,\ldots,r-1-\lceil\log_p k\rceil\}}(x_1, x_2, \ldots, x_r).$$

(Proof) By Theorem 3.1 and Lemma 4.1, f is k-undecomposable for these bipartitions. □

Example 4.4 *Consider the case where $p = 2$, $n = 5$, $k = 1$ and $r = 3$. If $f(x_1, x_2, x_3, 0, 0)$ is totally 1-undecomposable, then f is undecomposable for the bipartitions*

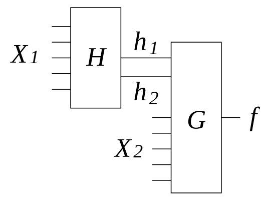
$$u = S^3_2(x_1, x_2, x_3) = \bar{x}_1 x_2 x_3 \vee x_1 \bar{x}_2 x_3 \vee x_1 x_2 \bar{x}_3.$$

Note that u denotes the same set of bipartitions as Example 3.4. ∎

Example 4.5 *Suppose that we have to check whether the given 2-valued 10-variable function $f(x_1, x_2, \ldots, x_{10})$ can be realized by a network shown in Fig. 4.1. In this case, the straightforward method needs to check all possible bipartitions (X_1, X_2), where $|X_1| = 5$ and $|X_2| = 5$. This set of bipartitions is represented by $S^{10}_5(x_1, x_2, \ldots, x_{10})$, and the number of bipartitions to consider is $|S^{10}_5(x_1, x_2, \ldots, x_{10})| = C(10,5) = 252$. However, if $f(x_1, x_2, x_3, x_4, x_5, 0, 0, 0, 0, 0)$ is totally 2-undecomposable, then we need not check for $C(5,3) \times C(5,2) = 100$ bipartitions. This fact is explained as follows: From Theorem 4.1, the set of bipartitions that will not produce 2-decomposition is given by*

$$u_1 = S^5_3(x_1, x_2, x_3, x_4, x_5) S^5_2(x_6, x_7, x_8, x_9, x_{10}).$$

In u_1, the first factor selects three variables from $\{x_1, x_2, x_3, x_4, x_5\}$ as bound variables, and the second factor selects two variables from $\{x_6, x_7, x_8, x_9, x_{10}\}$ as bound variables. For example, suppose that $\{X_1\} = \{x_1, x_2, x_3, x_6, x_7\}$ is selected as a bound set, and $\{X_2\} = \{x_4, x_5, x_8, x_9, x_{10}\}$ is selected as a free set. This bipartition (X_1, X_2) does not produce 2-decomposition, since $\{x_1, x_2, x_3\}$ is in the bound set and $\{x_4, x_5\}$ is in the free



$$f(X_1, X_2) = g(h_1(X_1), h_2(X_1), X_2)$$

Figure 4.1: 2-decomposition of 10-variable function.

set, and $f(x_1, x_2, x_3, x_4, x_5, 0, 0, 0, 0, 0)$ is totally 2-undecomposable. Note that $|u_1| = C(5,3)C(5,2) = 10 \times 10 = 100$.

In a similar way, if $f(0, 0, 0, 0, 0, x_6, x_7, x_8, x_9, x_{10})$ is also totally 2-undecomposable, then the following bipartitions need not be checked:

$$u_2 = S^5_3(x_6, x_7, x_8, x_9, x_{10}) S^5_2(x_1, x_2, x_3, x_4, x_5).$$

u_2 denotes $C(5,3) \times C(5,2) = 100$ bipartitions for that no 2-decomposition exist. So, we need only to check for the following bipartitions:

$$bp = S^{10}_5(x_1, x_2, \ldots, x_{10}) \bar{u}_1 \bar{u}_2.$$

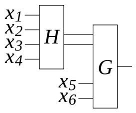
Since u_1 and u_2 are mutually disjoint, we have only to check

$$|bp| = |S^{10}_5(x_1, x_2, \ldots, x_{10})| - |u_1| - |u_2| \\ = C(10,5) - 100 - 100 = 52$$

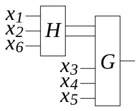
bipartitions. In this case, we can reduce the search space into one fifth by finding two subfunctions that are 2-undecomposable. ∎

Theorem 4.2 *Let $f(x_1, x_2, \ldots, x_{n-1}, a)$ be totally k-undecomposable, where $a \in P$. Then, $f(x_1, x_2, \ldots, x_{n-1}, x_n)$ is totally k-undecomposable iff f is undecomposable for the following $C(n-1, \lceil\log_p k\rceil) + C(n-1, k)$ bipartitions: $S^{n-1}_{\{n-1-\lceil\log_p k\rceil\}}(x_1, x_2, \ldots, x_{n-1})\bar{x}_n \vee S^{n-1}_k(x_1, x_2, \ldots, x_{n-1})x_n$.*

Example 4.6 *Consider the case where $n = 6$, $p = 2$, and $k = 2$. Suppose that $f(x_1, x_2, x_3, x_4, x_5, 0)$ is totally k-undecomposable. To show that f is totally undecomposable, we need the followings: For $x_6 = 0$, we have to check for the bipartitions denoted by $S^5_4(x_1, x_2, x_3, x_4, x_5)\bar{x}_6$. Fig. 4.2(a) shows an example of $C(5,4)$ bipartitions. For $x_6 = 1$, we have to check for bipartitions denoted by $S^5_2(x_1, x_2, x_3, x_4, x_5)x_6$. Fig. 4.2(b) shows an example of $C(5,2) = 10$ bipartitions.* ∎



(a) 2-decomposition corresponding to $\bar{x}_6$.



(b) 2-decomposition corresponding to x_6.

Figure 4.2:

V Number of Totally k-Undecomposable Functions

When n is sufficiently large, almost all functions are totally 1-undecomposable [19]. In this part, we will show that almost all functions are also totally k-undecomposable.

5.1 p-valued case

Lemma 5.1 *Let $N_d(n,p,k)$ be the number of n-variable p-valued k-decomposable functions. Then,*

$$N_d(n,p,k) \leq \sum_{n_1=k+1}^{n-1-\lceil \log_p k \rceil} C(n,n_1) p^{kp^{n_1}+p^{n-n_1+k}}.$$

(Proof) Suppose that a function f has a disjoint k-decomposition shown in Fig. 1.1. First, consider the module H. The number of ways to select bound variables is $C(n,n_1)$. Since H has n_1 inputs and k outputs, the number of functions for H is $p^{kp^{n_1}}$. Next, consider the module G. Since G has n_2+k inputs and single output, the number of functions for G is $p^{p^{n_2+k}} = p^{p^{n-n_1+k}}$. Hence, we have the lemma. □

Theorem 5.1 *Let $N_{ud}(n,p,k)$ be the number of n-variable p-valued totally k-undecomposable functions. Then,*

$$\frac{N_{ud}(n,p,k)}{p^{p^n}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

(Proof) Since $N_{ud}(n,p,k) + N_d(n,p,k) = p^{p^n}$, we will show that

$$\frac{N_d(n,p,k)}{p^{p^n}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (5.1)$$

From Lemma 5.1, we have

$$N_d(n,p,k) \leq \sum_{n_1=k+1}^{n-1-\lceil \log_p k \rceil} C(n,n_1) p^{kp^{n_1}+p^{n_2+k}}. \quad (5.2)$$

Since $C(n,n_1) < 2^n$, the right-hand-side of (5.2) is less than

$$2^n \sum_{n_1=k+1}^{n-1-\lceil \log_p k \rceil} p^{kp^{n_1}+p^{n_2+k}} = 2^n \sum_{n_1=k+1}^{n-1-\lceil \log_p k \rceil} p^{A(n_1)}, \quad (5.3)$$

where $A(n_1) = kp^{n_1} + p^{n_2+k}$. Note that $A(n_1)$ takes its maximum when $n_1 = k+1$ and $n_1 = n-1-\lceil \log_p k \rceil$, and the values of $A(n_1)$ are $kp^{k+1} + p^{n-1}$ and $p^{n-1} + kp^{k+1}$, respectively. Thus, $A(n_1) \leq p^{n-1} + C$, where C does not depend on n. So, (5.3) is less than

$$D(n) = 2^n (n-1-\lceil \log_p k \rceil - (k+1)) p^{p^{n-1}+C}.$$

Let us take the logarithm of $D(n)$, and we have

$$\log_p D(n) = n \log_p 2 + \log_p (n-k-\lceil \log_p k \rceil - 2) + p^{n-1} + C.$$

Since $\frac{\log_p D(n)}{p^n} \rightarrow \frac{1}{p}$ as $n \rightarrow \infty$, we can conclude that (5.1) holds. □

5.2 Two-valued case

When $n \geq 4$ and $p = 2$, most functions are totally 1-undecomposable. For $k = 1$, we obtained the values of $N_{ud}(n,p,k)$ by exhaustive enumeration:

$$N_{ud}(3,2,1) = 104,$$
$$N_{ud}(4,2,1) = 57,240, \text{ and}$$
$$N_{ud}(5,2,1) = 4,290,002,448.$$

When $n = 5$, there are $2^{2^5} = 2^{32} = 4,294,967,296$ functions. Thus, 99.9% of the functions are totally 1-undecomposable. The case of $k = 2$ is interesting, since some FPGAs have LUTs with two outputs [3]. The decompositions must satisfy the relation:

$$|X_1| \geq k+1 \text{ and } |X_2| \geq 1 + \lceil \log_2 k \rceil.$$

This requires that $|X| = |X_1| + |X_2| \geq k + 2 + \lceil \log_2 k \rceil$. Thus, when $k = 2$, only the functions with $n \geq 5$ are interesting. For $n = 5$, the only case is $n_1 = 3$ and $n_2 = 2$, and we have $N_{ud}(5,2,2) = 3,744,402,432$. Thus, 87.2% of the 5-variable functions are totally 2-undecomposable.

VI Conclusion and Comments

In this paper, we defined totally k-undecomposable logic functions, and showed a systematic method to find a set of bipartitions that will not produce disjoint k-decompositions. Key contributions are:

1) Generation of a set of k-undecomposable bipartitions from totally k-undecomposable subfunctions.
2) Representation of k-undecomposable bipartitions by an n-variable switching function.
3) Enumeration of totally k-undecomposable functions.

The presented method can be extended to the case of incompletely specified functions. This method can be combined to existing decomposition methods to reduce search space.

When $n = 3$ or 4, $p = 2$ and $k = 1$, totally k-undecomposable functions are easily detected by BDDs and look-up tables [17]. By using this method, we can show the undecomposability of randomly generated functions very quickly.

We decomposed more than four thousand benchmark functions including functions with 256 inputs and 245 outputs [16, 17]. Experimental results for $p = 2$ and $k = 1$ show that for 1-undecomposable functions, the computation time were reduced to up to one hundreds. Currently, we are developing a system for k-decompositions with $k = 2$.

Even if the given functions have two-valued inputs only, functional decompositions with multi-valued inputs seems to be useful. This is explained as follows: Suppose that a completely specified two-valued input function has a k-decomposition of the form $f(X_1, X_2) = g(h_1(X_1), g_2(X_1), \ldots, g_k(X_1), X_2)$, and that $\mu(f : X_1, X_2) < 2^k$. In this case, assigning $\mu(f : X_1, X_2)$ different binary vectors to the k outputs of H produces don't care conditions for function g. This makes decomposition problem very difficult [21]. However, if we do not assign the binary vectors to the output of h, but assume that h produces a multiple-valued output, then no don't cares are generated. In this case, the decomposition problem is easier.

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