Bounds on the Average Number of Products in the Minimal Sum-of-Products Expressions for Multiple-Valued Input Two-Valued Output Functions

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Abstract: A lower bound $L_p(n,u)$ and an upper bound $U_p(n,u)$ on $S_p(n,u)$ are derived, where $S_p(n,u)$ is the average number of the products in the minimum sum-of-products expression for p-valued input two-valued output functions, n is the number of the inputs, and u is the number of the input combinations which are mapped into 1. The values of $S_p(n,u)$ are obtained by minimizing randomly generated functions, and they are compared with the calculated values of $U_p(n,u)$ and $L_p(n,u)$. These bounds are useful for estimating the size of programmable logic arrays.

Index Terms: Sum-of-products expression, complexity, prime implicants, logic minimization, programmable logic array, multiple-valued logic.

1. Introduction

A p-valued input two-valued output function f is a mapping $f: P^n \rightarrow B$, where $P=\{0,1,\ldots,p-1\}$ and $B=\{0,1\}$. It is a generalization of an ordinary switching function $f: B^n \rightarrow B$. A programmable logic array (PLA) with r-bit decoders directly realizes a sum-of-products expression (SOPE) of a 2^r-valued input two-valued output function [SAS 81],[SAS 84]. Multiple-valued logic minimizers such as MINI [HON 74], MINI-II [SAS 84], ESPRESSO-MV [RUD 85], QM[SAS 86] and ESPRESSO-EXACT [RUD 86] are used to obtain the near minimum SOPE's for p-valued input two-valued output functions.

This paper deals with the average number of products in the minimum SOPE's for p-valued input two-valued output functions. Because it is very important in the complexity analysis of AND-OR two-level logic circuits, including standard PLA's and PLA's with two-bit decoders, many researchers have spent considerable efforts [PAP 78]. Cobham, Fridshal and North [COB 62] obtained the average number of the products in minimum SOPE's of switching functions of up to 8 variables by using computer simulation. Mileto and Putzolu [MIL 64] obtained formulas for the average number of prime implicants and essential prime implicants of switching functions: their formulas give upper and lower bounds on the average number of products in minimum SOPE's, respectively. Glagolev [GLA 70] obtained an upper bound on the number of products in minimum SOPE's for almost all switching functions. Cook and Flynn [COO 73] investigated the average minimum cost of SOPE's and attempted to relate it to the entropy function. The author derived the formula for the average number of prime implicants for p-valued case [SAS 79], and also obtained the average number of products in minimum SOPE's for p=2 and p=4 by using computer simulation [SAS 80,81]. Recently, Bender and Butler [BEN 86] improved the upper and lower bounds for switching functions.

In this paper, we derive an upper and a lower bound on the average number of products in the minimum SOPE's. They are tighter than any other bounds reported to date. In Section 2, we show basic properties of the p- valued input two-valued output functions. In Section 3, we derive an upper bound on the average number of products in minimum SOPE's. In Section 4, we derive a lower bound on the average number of products in minimum SOPE's. In Section 5, we show the experimental results.

2. Definitions and Basic Properties

Definition 2.1: Let $P=\{0,1,\ldots,p-1\}$ be a set of truth values, and X be a variable which takes a value in P. Let S be a subset of P, then X^S denotes a 2-valued function $P \rightarrow B$ such that

$$X^S = \begin{cases} 0 & (\text{when } X \notin S) \\ 1 & (\text{when } X \in S) \end{cases},$$

where $B=\{0,1\}$. A symbol X^S is called a literal.

Definition 2.2: A product of literals is called a product term (or product), and a sum of products is called a sum-of-products expression (SOPE).

Lemma 2.1: An arbitrary p-valued input two-valued output function $f: P^n \rightarrow B$ can be represented by the following SOPE:

$$f(X_1,X_2,\ldots,X_n)= \bigvee_{(S_1,S_2,\ldots,S_n)} X_1^{S_1}\cdot X_2^{S_2}\cdot\ldots\cdot X_n^{S_n},$$

where $S_i \subseteq P$, $(i=1,2,\ldots,n)$.

A p-valued input two-valued output function is sometimes simply called a function.

Definition 2.3: An SOPE which represents f is said to be minimum if the SOPE has the minimum number of products. The number of the products in the minimum SOPE of f is denoted by t(f).

Definition 2.4: The set of inputs which are mapped into 1 by a function f is denoted by $f^{-1}(1)$. The number of elements in $f^{-1}(1)$ is called a weight of f and denoted by $|f|$. The average number of products in minimum SOPE's for n-variable p-valued input two-valued output functions with weight u is denoted by $S_p(n,u)$.

Lemma 2.2: $S_p(n,u) \leq \mathrm{Min}(u, p^{n-1})$.

(Proof)

1) An arbitrary function with weight u can be represented by an SOPE:

$$f(X_1,X_2,\ldots,X_n)= \bigvee_{\underline{a}} X_1^{a_1}X_2^{a_2}\cdot\ldots\cdot X_n^{a_n},$$

where the logical sum is taken for all input combinations $\underline{a}=(a_1,a_2,\ldots,a_n)$ such that $f(\underline{a})=1$. There are u such combinations and so, we have $t(f) \leq u$.

2) An arbitrary function can be represented by an SOPE:

$$f(X_1,X_2,\dots,X_n)= \bigvee_{\underline{b}} g_{\underline{b}}(X_1)X_2^{b_2}X_3^{b_3}\cdot\dots\cdot X_n^{b_n} ,$$

----------(2.1)

where the logical sum is taken for all input combinations of $\underline{b}=(b_1,b_2,\dots,b_n)$ in P^{n-1}. Therefore, we have $t(f)\le p^{n-1}$.

From 1) and 2), we have the lemma. (Q.E.D.)

Definition 2.5: A map of an n-variable p-valued input two-valued output function consists of p^n cells. Cells that contain 1's are called 1-cells while cells that contain 0's are called 0-cells.

Example 2.1: Fig.2.1 shows a map of a four-valued input two-valued output function. The SOPE for this function having the form (2.1) is

$$f(X_1,X_2)=X_1^{\{0,1,2,3\}}X_2^{\{0\}}\vee X_1^{\{0,2,3\}}X_2^{\{1\}}$$
$$\vee\ X_1^{\{0,1,3\}}X_2^{\{2\}}\vee X_1^{\{1\}}\cdot X_2^{\{3\}}.$$

(End of example).

X_2 \ X_1	0	1	2	3
0	1	1	1	1
1	1		1	1
2	1	1		1
3		1		

Fig.2.1 4-valued input two-valued output function

Mileto and Putzolu [MIL 64] derived formulas for $G_2'(n,u)$, the average number of prime implicants of switching functions with weight u, and $G_2^{*}(n,u)$, the average number of essential prime implicants of switching functions with weight u. Because

$$G_2^{*}(n,u)\ \le\ S_2(n,u)\ \le\ G_2'(n,u),$$

$G_2'(n,u)$ and $G_2^{*}(n,u)$ are upper and lower bounds on $S_2(n,u)$, respectively. Unfortunately, when $u\ge 2^{n-1}$ and $n\ge 10$, $G_2'(n,u)$ is greater than 2^{n-1} and $G_2^{*}(n,u)$ is very small compared with $S_2(n,u)$. Therefore, in such cases, these bounds give little information about the value of $S_2(n,u)$.

3. Upper Bound on the Average Number of Products in Mininum SOPE's

In this section, we derive $U_p(n,u)$, the upper bound on the average number of products in minimum SOPE's for n-variable p-valued input two-valued output functions with weight u.

There are $F^{(u)}=\binom{w}{u}$ different functions with weitht u, where $w=p^n$. Let $f_i^{(u)}$ be the i-th function with weight u ($i=1,2,\dots,F^{(u)}$). By the definition of $S_p(n,u)$, we have

$$S_p(n,u)=\frac{1}{F^{(u)}}\sum_{i=1}^{F^{(u)}} t(f_i^{(u)}) .\qquad \text{------}(3.1)$$

Lemma 3.1: An arbitrary n-variable function f can be represented by an expression:

$$f(X_1,X_2,\dots,X_n)= \bigvee_{\underline{a}\in P^{n-k}} E(\underline{a}),$$

where $E(\underline{a})=g_{\underline{a}}(X_1,X_2,\dots,X_k)\cdot X_{k+1}^{a_{k+1}}X_{k+2}^{a_{k+2}}\cdot\dots\cdot X_n^{a_n}$

$g_{\underline{a}}(X_1,X_2,\dots,X_k)=f(X_1,X_2,\dots,X_k,a_{k+1},\dots,a_n)$,

$\underline{a}=(a_{k+1},a_{k+2},\dots,a_n)$, and $a_j\in P$ ($j=k+1,\dots,n$).

$E(\underline{a})$ in Lemma 3.1 is called an E-term.

Example 3.1: The function shown in Fig.3.1 can be represented by an expression:

$f(X_1,X_2,X_3,X_4)=E(0,0)\vee E(0,1)\vee E(1,0)\vee E(1,1)$,

where $E(0,0)=(\ \ 1\ \ )\bar{x}_3\bar{x}_4$,

$E(0,1)=(x_1\vee\bar{x}_2)\bar{x}_3x_4$,

$E(1,0)=(\bar{x}_1x_2)x_3\bar{x}_4$, and

$E(1,1)=(\bar{x}_1\vee\bar{x}_2)x_3x_4$. (End of Example)

By Lemma 3.1, $f_i^{(u)}$ in (3.1) can be represented as

$$f_i^{(u)}= \bigvee_{\underline{a}\in P^{n-k}} E_i(\underline{a}) .$$

Therefore, we have

$$t(f_i^{(u)})\le \sum_{\underline{a}\in P^{n-k}} t(E_i(\underline{a}))\quad \text{-------------}(3.2)$$

(X_3,X_4) \ (X_1,X_2)	00	01	11	10
00	1	1	1	1
01	1		1	1
11	1	1		1
10		1		

Fig.3.1 A 4-variable switching function

Example 3.2: In the expression of Example 3.1, $t(E(0,0))=1$, $t(E(0,1))=2$, $t(E(1,0))=1$, and $t(E(1,1))=2$. Therefore, the right hand side of (3.2) is equal to $\sum_{\underline{a}\in P^2} t(E(a))=1+2+1+2=6$. On the other hand, the left hand side of (3.2) is $t(f)=4$ as shown in Fig.3.2. Therefore, the relation (3.2) holds in this example. (End of example).

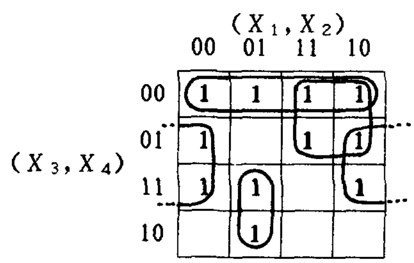

Fig.3.2 Minimum SOPE for Fig.3.1

By (3.1) and (3.2), we have $S_p(n,u) \leq U_p(n,u)$, where

$$U_p(n,u)=\frac{1}{F^{(u)}}\sum_{i=1}^{F^{(u)}}\sum_{\underline{a}\in P^{n-k}} t(E_i(\underline{a})).$$

By changing the order of summation, we obtain

$$U_p(n,u)=\frac{1}{F^{(u)}}\sum_{\underline{a}\in P^{n-k}}\sum_{i=1}^{F^{(u)}} t(E_i(\underline{a})).$$

Note that

$$\sum_{i=1}^{F^{(u)}} t(E_i(\underline{a})) \text{ -----------------(3.3)}$$

denotes the average number of products in minimum SOPE's for the functions with weight u having

$E_i(\underline{a})=g_{i\,\underline{a}}(X_1,X_2,\ldots,X_k)\cdot X_{k+1}^{a_{k+1}}\cdot X_{k+2}^{a_{k+2}}\cdot\ldots\cdot X_n^{a_n}$

as E-terms. The value of (3.3) does not depend on $\underline{a}$. $g_{i\,\underline{a}}(X_1,X_2,\ldots,X_k)$,$(i=1,2,\ldots,F^{(u)})$ are k-variable functions and have p^{p^k} different patterns.

Let $[F^k]$ be the set of all the k-variable functions. (3.3) can be represented by the sum with respect to the functions $g_j\in[F^k]$ and

$$U_p(n,u)=\frac{p^{n-k}}{F^{(u)}}\cdot\sum_{g_j\in[F^k]} t(E(g_j))\cdot L_p(n,g_j,u),$$

where $E(g_j)=g_j(X_1,X_2,\ldots,X_k)X_{k+1}^{a_{k+1}}X_{k+2}^{a_{k+2}}\cdot\ldots\cdot X_n^{a_n}$.
$L_p(n,g_j,u)$ denotes the number of n-variable functions with weight u having $E(g_j)$ as an E-term. Let c_0 be an E-term. The volume of c_0 is $|g_j|$.
$L_p(n,g_j,u)$ is equal to the number of different functions with weight u, where $|g_j|$ 1-cells and $(p-|g_j|)$ 0-cells are fixed.

Therefore, $L_p(n,g_j,u)=\binom{w-p^k}{u-|g_j|}$.

Because $t(E(g_j))=t(g_j)$, we have

$$U_p(n,u)=\frac{p^{n-k}}{F^{(u)}}\cdot\sum_{g_i\in[F^k]} t(g_j)\cdot\binom{w-p^k}{u-|g_j|} .$$

<u>Definition 3.1:</u> The relation $\sim$ satisfying the following conditions is called <u>VP-equivalence relation</u>.

1) $f \sim f$.
2) If $f_1=f(\ldots,X_i,\ldots,X_j,\ldots)$ and $f_2=f(\ldots,X_j,\ldots,X_i,\ldots)$, then $f_1 \sim f_2$. (Permutation of input variables)
3) Let $\sigma:P\rightarrow P$ be an arbitrary one-to-one mapping. If $f_1=f(\ldots,X_i,\ldots)$ and $f_2=f(\ldots,\sigma(X_i),\ldots)$ then $f_1 \sim f_2$.
(Permutation of values in a variable)

Especially, when p=2, VP-equivalence is called NP-equivalence [HAR 65],[MUR 79].

By using VP-equivalence relation, we can partition $[F^k]$ into equivalence classes.

Let $g_1,g_2,\ldots,g_\lambda$ be the representative functions of the VP-equivalence classes. Then, we have

$$U_p(n,u)=\frac{p^{n-k}}{F^{(u)}}\cdot\sum_{j=1}^{\lambda} t(g_j)\cdot\mu(g_j)\cdot\binom{w-p^k}{u-j} ,$$

where $\mu(g_i)$ is the number of functions which are VP-equivalent to g_i.

Next, by classifying λ different equivalence classes by the weight of the functions, we have

$$U_p(n,u)=\frac{p^{n-k}}{F^{(u)}}\sum_{j=1}^{p^k} c(j)\binom{w-p^k}{u-j} ,$$

where $c(j)=\sum_{|g_i|=j}\mu(g_j)\cdot t(g_i)$

and $g_1, g_2,\ldots, g_\lambda$ are representative functions of VP-equivalence classes. Finally, we have the following:

<u>Theorem 3.1:</u> Let $U_p(n,u)$ be an upper bound on the average number of the products in minimum SOPE's for n-variable p-valued input two-valued output functions. Then

$$U_p(n,u)=\frac{p^{n-k}}{F^{(u)}}\sum_{j=1}^{p^k} c(j)\cdot\binom{w-p^k}{u-j} ,$$

where $c(j)=\sum_{|g_i|=j}\mu(g_j)\cdot t(g_i)$,

$g_1,g_2,\ldots,g_\lambda$ are representative functions of VP-equivalence classes, $t(g_i)$ is the number of products in a minimum SOPE for g_i, and $\mu(g_i)$ is the number of functions which are VP-equivalent to g_i.

<u>Example 3.3:</u> Suppose that p=2 and k=3. The coefficients c(j) can be obtained as follows: Table 3.1 shows the representative functions of VP (=NP)-equivalence classes of three variables. There are 22 classes. By minimizing all the representative functions, we have the coefficients shown in Table 3.2. (End of example).

<u>Example 3.4</u>: Suppose that p=2 and k=4.
There are 402 different VP-equivalence classes of 4-variable functions. In a similar way, we have the coefficients c(j) shown in Table 3.3. Logic minimization of the functions was done by QM [SAS 86], a modified Quine-McCluskey algorithm for p-valued input two-valued output functions.(End of example).

<u>Example 3.5:</u> Suppose that p=4 and k=2.
There are 192 different representative functions of four-valued input two-valued output functions. Table 3.4 shows the coefficients c(j). Logic minimization of the functions was done by QM. (End of example).

4. <u>Lower Bound on the Average Number of Products in Minimum SOPE's</u>.

In this section, we derive $L_p(n,u)$, the lower bound on the average number of products in minimum SOPE's for n-variable p-valued input two-valued output functions with weight u.

<u>Definition 4.1:</u> A product $p_1=X_1^{S_1}X_2^{S_2}\cdot\ldots\cdot X_n^{S_n}$ is an implicant of f if $p < f$. A product p_1 is said to be a <u>prime implicant</u> of f if there is no product p_2 such that $p_2 < f$, $p_1 < p_2$, and $p_1 \neq p_2$.
Let there be k_j different S_i's such that $|S_i|=j$ for $j=1,2,\ldots,p$. Then, this product is said to be a <u>k-cube</u>, where $\underline{k}=(k_1,k_2,\ldots,k_p)$.

<u>Example 4.1:</u> Let p=4 and n=4.
$X_1^{\{0,1,3\}}\cdot X_2^{\{0,1\}}\cdot X_3^{\{1,2\}}\cdot X_4^{\{0\}}$ is a (1,2,1,0)-cube, while
$X_1^{\{0,1,2,3\}}\cdot X_2^{\{1,2\}}\cdot X_3^{\{2,3\}}\cdot X_4^{\{0,1,2\}}$ is a (0,2,1,1)-cube.

Table 3.1 Representative Functions of 3 variables

	Representative Function g_j	Weight $\lvert g_j \rvert$	Number of Functions in the class $\mu(g_j)$	Number of Products in Minimum SOPE $t(g_j)$
1	0	0	1	0
2	$\bar{a}\bar{b}\bar{c}$	1	8	1
3	$\bar{a}\bar{b}$	2	12	1
4	$\bar{a}(\bar{b}c \vee b\bar{c})$	2	12	2
5	$\bar{a}(\bar{b} \vee \bar{c})$	3	24	2
6	$\bar{a}$	4	6	1
7	$\bar{a}bc \vee \bar{a}b\bar{c} \vee a\bar{b}\bar{c}$	3	8	3
8	$\bar{a}\bar{b} \vee \bar{b}\bar{c} \vee \bar{c}\bar{a}$	4	8	3
9	$a\bar{b}\bar{c} \vee \bar{a}bc$	2	4	2
10	$\bar{a}bc \vee \bar{b}\bar{c}$	3	24	2
11	$\bar{a}c \vee \bar{b}\bar{c}$	4	24	2
12	$\bar{a}b \vee \bar{a}c \vee a\bar{b}\bar{c}$	4	24	3
13	$\bar{a} \vee \bar{b}\bar{c}$	5	24	2
14	$\bar{a}b \vee a\bar{b}$	4	6	2
15	$\bar{a}b \vee a\bar{b} \vee \bar{a}\bar{c}$	5	24	3
16	$\bar{a} \vee \bar{b}$	6	12	2
17	$a \oplus b \oplus c \oplus 1$	4	2	4
18	$\bar{a}c \vee \bar{a}\bar{b} \vee \bar{b}c \vee ab\bar{c}$	5	8	4
19	$\bar{a} \vee \bar{b}c \vee b\bar{c}$	6	12	3
20	$a\bar{b} \vee b\bar{c} \vee \bar{a}c$	6	4	3
21	$\bar{a} \vee \bar{b} \vee \bar{c}$	7	8	3
22	1	8	1	1

Table 3.2 c(j) for p=2 and k=3

j	1	2	3	4	5	6	7	8
c(j)	8	44	120	170	152	72	24	1

Table 3.3 c(j) for p=2 and k=4

j	1	2	3	4	5	6	7	8
c(j)	16	208	1328	5288	14720	29872	46368	54992
j	9	10	11	12	13	14	15	16
c(j)	50992	36336	19856	8056	2352	448	64	1

Table 3.4 c(j) for p=4 and k=2

j	1	2	3	4	5	6	7	8
c(j)	16	192	1184	4508	12336	24248	36992	42720
j	9	10	11	12	13	14	15	16
c(j)	37072	24632	13056	5120	1360	240	32	1

Definition 4.2: $G'_p(n,\underline{k},u)$ denotes the average number of prime $\underline{k}$-cubes of n-variable p-valued input two-valued output functions with weight u.

Theorem 4.1:

$$G'_p(n,\underline{k},u) = \frac{C^{(\underline{k})}}{F^{(u)}} \cdot \sum_{t=0}^{\eta(\underline{k})} (-1)^t \cdot \sum_{\underline{t}} \lambda(\underline{k},\underline{t}) \cdot \binom{w-w(\underline{k},\underline{t})}{u-w(\underline{k},\underline{t})},$$

where $F^{(u)}=\binom{w}{u}$, $w=p^n$,

$$C^{(\underline{k})}=(n!)\cdot \prod_{i=1}^{p} \left[\frac{1}{k_i!} \binom{p}{i}^{k_i}\right],$$

$$w(\underline{k},\underline{t})=w(\underline{k})\cdot\left(1+\sum_{i=1}^{p-1} \frac{t_i}{i}\right),$$

$$w(k)=\prod_{i=1}^{p} (i)^{k_i},\quad \eta(\underline{k})=\sum_{i=1}^{p-1} a_i,\quad a_i=k_i(p-1),$$

$\underline{t}=(t_1,t_2,\ldots,t_{p-1})$ is a partition of t, and

$$\lambda(\underline{k},\underline{t})=\prod_{i=1}^{p-1} \binom{a_i}{i}.$$

(Proof is shown in [SAS 79a].)

Lemma 4.1: Let $V_p(n,u)$ be the average volume of prime cubes. Then,

$V_p(n,u) = A/B$, where

$$A = \sum_{\underline{k}} w(\underline{k})\cdot G'_p(n,\underline{k},u), \text{ and } B = \sum_{\underline{k}} G'_p(n,\underline{k},u)$$

(Proof) It is easy to see that A denotes the sum of the average volume of the prime cubes, and that B denotes the average number of prime cubes. Hence, A/B denotes the average volume of prime cubes. (Q.E.D.)

Now, we will make the following assumption.

Assumption 4.1: The average volume of prime cubes in a minimal SOPE for f is equal to the average volume of all the prime cubes of f.

(Note that, in general, there are many minimum SOPE's for a function f.)

By using Assumption 4.1, we have the following:

Conjecture 4.1: $S_p(n,u) \geq L_p(n,u)$,

where $L_p(n,u)=(u\cdot B)/A$,

$$A = \sum_{\underline{k}} w(\underline{k})\cdot G'_p(n,\underline{k},u), \text{ and } B = \sum_{\underline{k}} G'_p(n,\underline{k},u)$$

(Explanation supporting the conjecture)

There are u minterms in f. Because the average volume of each cube is $V_p(n,u)$, any SOPE for f requires at least $u/V_p(n,u)$ cubes to cover all the minterms of f. (End of the explanation)

Example 4.2: Consider the function shown in Fig.3.1, where n=4, p=2, and u=11. The number of prime cubes is 7, and the sum of volumes of all prime cubes is 22. The average volume of prime cubes is 22/7=3.14 . If Assumption 4.1 holds, then the lower bound on the number of products in minimum SOPE for f is 11/3.14=3.5. Fig.3.2 shows a minimum SOPE for f. Note that only two prime implicants are essential. Therefore, Assumption 4.1 gives a tighter lower bound than the bound given by the number of essential prime implicants. (End of example).

5. Experimental Results

In order to obtain $S_p(n,u)$, functions generated by a pseudo-random number generation algorithm+ were minimized. For each value of density, $d=\frac{u}{2^n}$, n and p, 100 sample functions were generated for (p=2, n=6 and 8, and p=4, n=3 and 4), and 10 samples were generated for (p=2, n=10 and p=4, n=5). Then, each function was minimized by QM.

Table 5.1 shows the values of $S_2(n,u)$ obtained by the average of the sample functions, as well as calculated values of $G'_2(n,u)$, $G^{\bullet}_2(n,u)$, $U_2(n,u)$, and $L_2(n,u)$ for n=6, 8, and 10. As easily seen from this table, $U_2(n,u)$ and $L_2(n,u)$ are, in most cases, tighter bounds than $G'_2(n,u)$ and $G^{\bullet}_2(n,u)$, respectively. Table 5.2 shows the case of p=4, and for n=3,4, and 5. In this case, however, $G^{\bullet}_4(n,u)$ is omitted because no formula is known for it. Values of $U_4(n,u)$ were calculated by using the coefficients shown in Tables 3.3 and 3.4.

6. Conclusion

In this paper, we derived an upper bound $U_p(n,u)$ and a lower bound $L_p(n,u)$ on $S_p(n,u)$, the average number of prime implicants in minimum SOPE's for n-variable p-valued input two-valued output functions using u, the weight of the functions, as a parameters. $U_p(n,u)$ was derived by using the statistical data of minimum SOPE's for all k-variable functions ($k \leq n$). On the other hand, $L_p(n,u)$ was derived by using the concept of the average volume of the prime cubes. These bounds are tighter than any other bounds reported to date, and applicable to any value of p.

+ N-variable pseudo-random functions were generated as follows [YAI 86]:

```
R=1              (random number initial value)
table(1,2,...,2**N)=0 (reset the truth table)
do the followings until the density becomes  the
specified value.
  R=MOD(163*R+656329,12518383)
  ADDR=MOD(R,2**N)
  table (ADDR)=1
end
```

Table 5.1 Average Numbers of the Prime Implicants in Minimum SOPE's and Their Upper and Lower Bounds (p=2)

$p = 2,\ n = 6$

$d=u/2^n$	1/8	2/8	3/8	4/8	5/8	6/8	7/8
u	8	16	24	32	40	48	56
$G'_2(n,u)$	6.41	12.45	18.61	24.15	29.16	31.13	26.52
$G''_2(n,u)$	6.11	8.86	8.87	7.33	5.34	3.60	2.76
$L_2(n,u)$	5.67	8.52	10.21	10.59	10.14	8.74	6.23
$U_2(n,u)$	6.59	11.04	14.47	16.68	17.63	17.16	14.97
$S_2(n,u)$	6.14	10.32	12.42	13.76	13.77	12.78	10.11

$p = 2,\ n = 8$

$d=u/2^n$	1/8	2/8	3/8	4/8	5/8	6/8	7/8
u	32	64	96	128	160	192 *	224 *
$G'_2(n,u)$	25.68	53.99	84.11	117.53	151.81	176.97	167.89
$G''_2(n,u)$	21.74	26.41	20.94	12.84	6.42	2.67	1.06
$L_2(n,u)$	20.08	30.17	34.96	35.41	33.26	27.20	17.96
$U_2(n,u)$	26.09	44.19	57.75	66.28	70.05	68.23	58.97
$S_2(n,u)$	22.91	35.99	43.25	45.83	44.13	39.01	28.49

$p = 2,\ n = 10$

$d=u/2^n$	1/8	2/8	3/8	4/8	5/8	6/8	7/8
u	128 *	256 *	384 *	512 *	640 *	768	896
$G'_2(n,u)$	105.76	237.23	387.31	584.72	800.91	1039.0	1138.1
$G''_2(n,u)$	77.13	76.48	46.31	20.27	6.60	1.53	0.00
$L_2(n,u)$	73.93	110.97	124.75	126.03	115.38	92.67	58.08
$U_2(n,u)$	103.92	177.98	230.62	264.69	279.73	272.70	234.94
$S_2(n,u)$	86.1	133.8	154.9	158.8	156.3		

n: number of the input variables
u: number of the minterms which are mapped into one.
$G'_p(n,u)$: average number of the prime implicants
$G''_p(n,u)$: average number of the essential prime implicants
$L_p(n,u)$: lower bound on $S_p(n,u)$
$U_p(n,u)$: upper bound on $S_p(n,u)$
$S_p(n,u)$: average number of prime the implicants in the minimum SOPE's
(The values of $S_p(n,u)$ were obtained by taking the average of ten randomly generated functions)
*: near minimal solution

Table 5.2 Average Numbers of the Prime Implicants in Minimum SOPE's and Their Upper and Lower Bounds (p=4)

$p=4,\ n=3$

$d=u/4^n$	1/8	2/8	3/8	4/8	5/8	6/8	7/8
u	8	16	24	32	40	48	56
$G'_4(n,u)$	5.94	11.69	17.88	24.24	29.67	32.01	25.65
$L_4(n,u)$	4.92	6.98	7.87	7.90	7.21	5.87	3.90
$U_4(n,u)$	6.01	9.43	11.70	12.59	12.17	10.70	8.37
$S_4(n,u)$	5.47	8.54	9.67	10.14	9.55	8.07	5.77

$p=4,\ n=4$

$d=u/4^n$	1/8	2/8	3/8	4/8	5/8	6/8	7/8
u	32	64	96	128 *	160 *	192 *	224 *
$G'_4(n,u)$	24.86	54.76	91.28	136.18	187.32	237.00	240.00
$L_4(n,u)$	17.17	24.49	27.01	26.59	23.59	18.51	11.36
$U_4(n,u)$	23.71	37.68	46.51	49.89	48.34	42.56	33.31
$S_4(n,u)$	20.29	29.72	33.57	33.52	31.32	26.69	17.92

$p=4,\ n=5$

$d=u/4^n$	1/8	2/8	3/8	4/8	5/8	6/8	7/8
u	128 *	256 *	384 *	512 *	640	768	896
$G'_4(n,u)$	107.39	256.57	464.78	756.63	1160.5	1695.9	2203.0
$L_4(n,u)$	63.32	89.76	97.70	95.24	83.26	63.87	37.37
$U_4(n,u)$	94.41	151.47	185.61	199.09	193.00	170.23	133.0
$S_4(n,u)$	74.5	106.6	118.2	121.6			

*: near minimal solution

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