ON THE COMPLEXITY OF SHALLOW LOGIC FUNCTIONS AND THE ESTIMATION OF PROGRAMMABLE LOGIC ARRAY SIZE

Tsutomu Sasao and Hiroaki Terada

Department of Electronic Engineering
Osaka University, Osaka 565, Japan

Abstract

A function $f:\{0,1,\ldots,p-1\}^m \to \{0,1\}$ is said to be shallow if $|f^{-1}(1)|\cdot(m\cdot\log_2 p) \le p^m$. Approximation formulas for the average number of prime implicants and essential prime implicants of the shallow functions are derived for p=2 and p=4. These formulas estimate the average size of progrmmable logic arrays for the shallow logic functions; the estimation needs only a pocket calculator. Experimental results and calculated results are compared.

I. Introduction

The number of basic elements used to realize a given function, i.e., the complexity of the logic function, is an important problem not only theoretically, but also practically[1]-[2].

The advent of programmable logic arrays (PLA's) has made two-level logic circuits important again. Glagolev[3] has shown that

$$2^{n-1}/(\log_2 n)\cdot(\log_2\log_2 n)$$

terms are necessary to represent almost all functions of n variables. Mileto and Putzolu[4] have derived formulas for the average number of prime implicants and essential prime implicants when the number of 1's in a truth table is known. Cobham, Fridshal, and North[5] have obtained the average number of prime implicants, essential prime implicants, and terms in a minimal sum-of-products expression by computer simulation. We have obtained the average size of PLA's for randomly generated functions of up to 10 variables by computer simulation[6].

In this paper, we will derive approximation formulas for the average number of prime implicants and essential prime implicants of shallow binary functions. A binary function is a two-valued function of multiple-valued variables[8]. A shallow function is a function of small weight, where the weight of the function f is defined as the number of 1's in the truth table. By using the formulas obtained in this paper, we can easily estimate the size of PLA's for shallow logic functions; the estimation needs only a pocket calculator.

In II, basic ideas of the estimation of the PLA size are illustrated.

In III, a design theory for the minimization of the AND array is summarized.

In IV, formulas for the number of prime implicants and essential prime implicants of binary functions are derived.

In V, some properties of shallow binary functions are described and approximation formulas for the average number of prime implicants and essential prime implicants are derived.

In VI, experimental results and calculated results are compared.

Proofs are omitted form this paper because of the space limitation.

II. Estimation of the Size of a Programmable Logic Array

In this paper, two types of PLA's are considered: two-level PLA's and PLA's with two-bit decoders. The first type of PLA, a two-level PLA, consists of an AND array and an OR array. In the AND array, products of input variables are generated; and in the OR array, sums of products are generated. Fig.2.1 shows a two-level PLA which realizes the function of Table 2.1. The second type of PLA, a PLA with two-bit decoders, consists of decoders, an AND array, and an OR array. Fig.2.2 shows a two-bit decoder, which generates all the maxterms of its input variables. Fig.2.3 shows a PLA with two-bit decoders which realizes the function of Table 2.1.

As we will show in III, the width W of the AND array of the PLA for the given function f is equal to the number of terms in the expression which represents f. In the case of the two-level PLA, the expression represents a function

$$f:\{0,1\}^n \to \{0,1\},$$

i.e., an ordinary boolean function of n-variables; whereas, in the case of the PLA with two-bit decoders, the expression represents a function

$$f:\{00,01,10,11\}^{n/2} \to \{0,1\},$$

i.e., a binary function of n/2 variables, where each variable takes four values. Therefore, in order to minimize the size of the PLA, we have to minimize the expression which represents the given function.

Many methods are known for the minimization of logical expressions, but when the number of variables is large, it becomes a difficult problem: we need a large computer to solve it[9]-[13]. In practical applications, we often want to estimate the size of the PLA before actually minimizing the given expression. For example, consider a PLA which realizes an eight-variable function of weight 32. Let W denote the width of the AND array. Then, it is easy to see that $1 \le W \le 32$. However, the functions for W=1 and W=32 are special ones, and in most cases, $2 \le W \le 31$. Then how many terms are necessary to realize most functions? To obtain the average width of the PLA's, we randomly generated

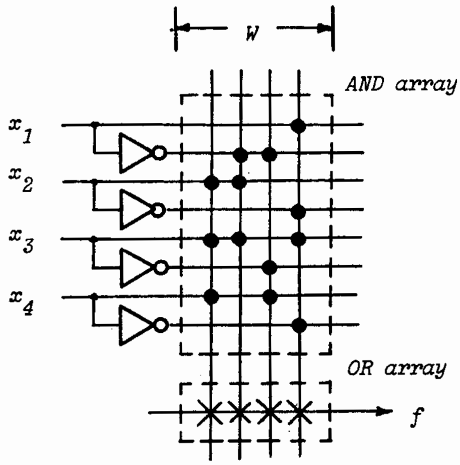

Fig.2.1 Two-level PLA for Table 2.1

Table 2.1 Four-variable function

x_1	x_2	x_3	x_4	f
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

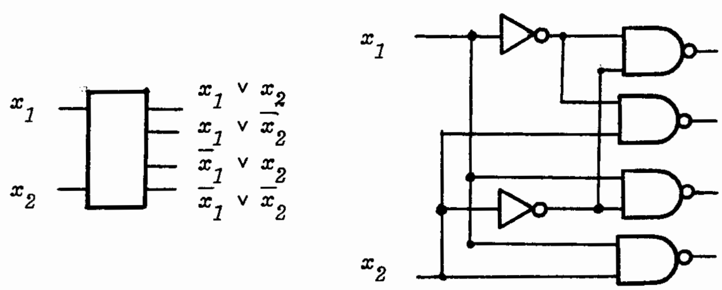

Fig.2.2 Two-bit decoder

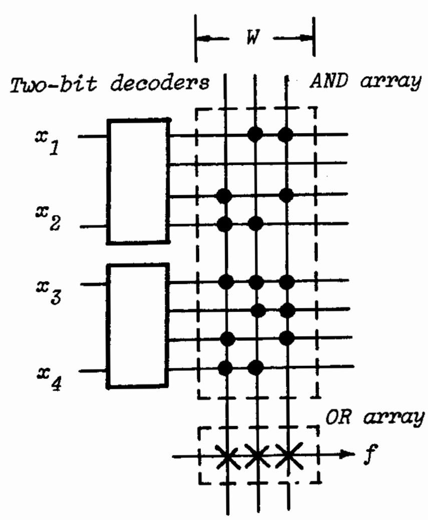

Fig.2.3 PLA with two-bit decoders for Table 2.1

10 functions of 8-variable whose weights are 32. Table 2.2 shows the distribution of the width of the PLA's. For each of these ten samples, the width of the two-level PLA is between 19 and 25, and the width of the PLA with two-bit decoders is between 18 and 23.

We also generated 8-variable functions of different weights. Fig.2.4 shows the average number of prime implicants, number of terms in a minimal sum-of-products expression(=W), and number of essential prime implicants. Fig.2.4 shows that most prime implicants are essential when the weight is smaller than 32.

For a PLA which realizes a given function, we have the following relation:

The number of essential prime implicants	$\leq$	The width of the PLA (=W)	$\leq$	The number of prime implicants

Therefore, we have

The average number of essential prime implicants	$\leq$	The average width of the PLA's	$\leq$	The average number of prime implicants

------(2.1)

As we will show in V, shallow binary functions have the following properties:

1) We can derive simple approximation formulas for the average number of prime implicants and essential prime implicants.
2) We can estimate the average number of terms in a minimal sum-of-products expression by obtaining the number of prime implicants and the number of essential prime implicants, because most prime implicants are essential.

In the case of the 8-variable functions, the condition for the shallowness implies that the weight is smaller than 32.

Therefore, we can easily estimate the average width of PLA's. With the formulas derived in V, the average width of PLA's for the 8-variable functions of weight 32 can be estimated as follows:

21.09	$\leq$	Average width of two-level PLA's	$\leq$	25.33	, and
17.54	$\leq$	Average width of PLA's with two-bit decoders	$\leq$	25.76	

These theoretical results agree with the experimental results in Table 2.2.

III. Minimization of the AND array

In this section, a design theory for minimization of the size of AND arrays is summarized [6]-[7].

Definition 3.1: Let $X=(x_1,x_2,\ldots,x_n)$ be a variable in B^n, where $B=\{0,1\}$. The set of variables in X is denoted by $\{X\}$. $(X_1,X_2,\ldots,X_r)$ is a partition of X iff $\{X_1\}\cup\{X_2\}\cup\ldots\cup\{X_r\}=\{X\}$, $\{X_i\}\cap\{X_j\}=\phi$ $(i\neq j)$, and $\{X_i\}\neq\phi$.

Definition 3.2: Let $\underline{a}=(a_1,a_2,\ldots,a_n)$ be a constant in B^n. A mapping

$$X^{\underline{a}} : B^n \rightarrow B$$

denotes a function such that $X^{\underline{a}} = 0$ if $X\neq\underline{a}$, and $X^{\underline{a}} = 1$ if $X=\underline{a}$. Let $S \subseteq B^n$. A symbol X^S denotes the function

$$X^S = \bigvee_{\underline{a}_i \in S} X^{\underline{a}_i} .$$

Example 3.1: Let $X=(x_1,x_2)$, $\underline{a}=(0,1)$ and $\underline{b}=(1,0)$.

$$X^{\underline{a}} = \begin{cases} 1 \text{ if } X=(0,1) \\ 0 \text{ if } X\neq(0,1) \end{cases}$$

$$X^{\{\underline{a},\underline{b}\}} = \begin{cases} 1 \text{ if } X=(0,1) \text{ or } X=(1,0) \\ 0 \text{ if } X=(0,0) \text{ or } X=(1,1) \end{cases}$$

$X^{\{(0,1),(1,0)\}}$ is sometimes denoted by $X^{\{01,10\}}$.

Definition 3.3: X^S is said to be a literal. A product of distinct literals is said to be a term. A sum of terms is said to be a sum-of-products expression. The number of the terms in a sum-of-products expression P is denoted by t(P). P is said to be minimal if there exists no expression Q such that $t(Q) < t(P)$ and that Q denotes the same function as P.

Theorem 3.1: Let $(X_1,X_2,\ldots,X_r)$ be a partition of X. An arbitrary function

$$f: B_1^{n_1} \times B_2^{n_2} \times \ldots \times B_r^{n_r} \rightarrow B$$

can be represented in a form

$$f(X_1,X_2,\ldots,X_r)= \bigvee_{(S_1,S_2,\ldots,S_r)} X_1^{S_1}\cdot X_2^{S_2}\cdot\ldots\cdot X_r^{S_r} , \quad \text{----(3.1)}$$

where $S_i \subseteq B^{n_i}$, $n_i=d(X_i)$ and $d(X_i)$ denotes the number of variables in $\{X_i\}$.

Theorem 3.2: In a PLA with decoders, if each decoder generates all the maxterms of {X} for $i=1,2,\ldots,r$, then an arbitrary term which has the form

$$X_1^{S_1}\cdot X_2^{S_2}\cdot\ldots\cdot X_r^{S_r}$$

can be realized in each column of the AND array.

Example 3.2: In the case of a two-level PLA. Let $X_1=(x_1)$, $X_2=(x_2)$, $X_3=(x_3)$, and $X_4=(x_4)$ be a (trivial) partition of $X=(x_1,x_2,x_3,x_4)$.

The function of Table 2.1 can be represented as follows:

$$f(X_1,X_2,X_3,X_4)=X_1^0X_2^0X_3^0X_4^1 \vee X_1^0X_2^1X_3^0X_4^1 \vee X_1^0X_2^1X_3^1X_4^0$$
$$\vee X_1^0X_2^1X_3^1X_4^1 \vee X_1^1X_2^0X_3^1X_4^0 \vee X_1^1X_2^1X_3^1X_4^1$$

This expression can be simplified as follows:

$$f(X_1,X_2,X_3,X_4)=X_1^{\{0,1\}}X_2^1X_3^1X_4^1 \vee X_1^0X_2^1X_3^1X_4^{\{0,1\}}$$
$$\vee X_1^0X_2^{\{0,1\}}X_3^0X_4^1 \vee X_1^1X_2^0X_3^1X_4^0 . \quad \text{-----(3.2)}$$

Fig.2.1 is a two-level PLA realization of this function. Each column of the AND array corresponds to each term of (3.2). Therefore, the number of columns of the AND array is equal to the number of the terms in (3.2).

Table 2.2 Distribution of the width of PLA's for 8-variable functions of weight 32.

Sample number	Two-level PLA	PLA with two-bit decoders
1	21	19
2	21	19
3	21	18
4	24	21
5	23	20
6	23	20
7	25	23
8	23	20
9	25	19
10	19	18
Average	22.5	19.7

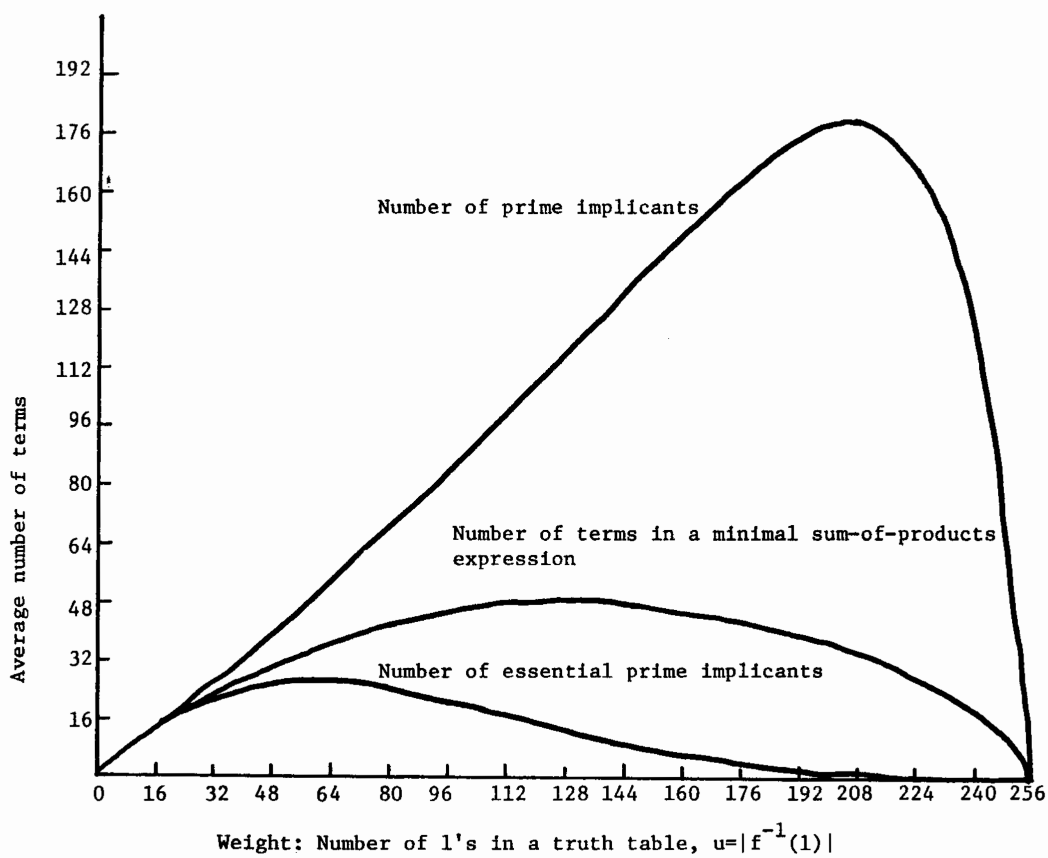

Fig.2.4 Average number of prime implicants, terms in a minimal sum-of-products expression, and essential prime implicants of 8-variable functions.

Example 3.3: In the case of a PLA with two-bit decoders.

Let $X=(X_1,X_2)$, $X_1=(x_1,x_2)$ and $X_2=(x_3,x_4)$ be a partition of $X=(x_1,x_2,x_3,x_4)$. The function of Table 2.1 can be represented as

$$f(X_1,X_2)=X_1^{(00)}\cdot X_2^{(01)} \vee X_1^{(01)}\cdot X_2^{(01)} \vee X_1^{(01)}\cdot X_2^{(10)} \vee X_1^{(01)}\cdot X_2^{(11)} \vee X_1^{(10)}\cdot X_2^{(10)} \vee X_1^{(11)}\cdot X_2^{(11)}$$

This expression can be simplified as follows:

$$f(X_1,X_2)=X_1^{\{00,01\}}\cdot X_2^{(01)} \vee X_1^{\{01,10\}}\cdot X_2^{(10)} \vee X_1^{\{01,11\}}\cdot X_2^{(11)}. \qquad ---(3.3)$$

Fig.2.3 is a PLA with two-bit decoders which realizes the function of Table 2.1. Each column of the AND array corresponds to each term of (3.3).

Theorem 3.3: By obtaining a minimal sum-of-products expression of f(X) which have the form (3.1), we can minimize the size of the AND array for f(X). In the case of the two-level PLA, the expression denotes the function

$$f:\ \{0,1\}^n \to \{0,1\},$$

whereas in the case of the PLA with two-bit decoders, the expression denotes the binary function of four-valued variables

$$f:\ \{00,01,10,11\}^r \to \{0,1\},$$

where $n=2r$.

IV. Average Number of Prime Implicants and Essential Prime Implicants of Binary Function

In this section, we derive formulas for the average number of prime implicants and essential prime implicants of binary functions. By using multiple-valued variables instead of two-valued variables, we can treat PLA's with decoders.

Definition 4.1: Let $P=\{0,1,\ldots,p-1\}$ and $B=\{0,1\}$. A function $f: P^m \to B$ is said to be a binary function. $u=|f^{-1}(1)|$, i.e., the number of 1's in the truth table, is said to be the weight of f. The fraction of 1's in the truth table, i.e., $d=u/p^m$ is said to be the density of f.

Lemma 4.1: A function $f: P^m \to B$ can be represented by the following expression.

$$f(X_1,X_2,\ldots,X_m)=\bigvee_{(S_1,S_2,\ldots,S_m)} X_1^{S_1}\cdot X_2^{S_2}\cdot\ldots\cdot X_m^{S_m},$$

where $S_i \subseteq P$, $X_i^{S_i}=0$ if $X_i \notin S_i$, and $X_i^{S_i}=1$ if $X_i \in S_i$.

Definition 4.2: A term which has the form $X_1^{S_1}\cdot X_2^{S_2}\cdots X_m^{S_m}$ is said to be an implicant of f if $X_1^{S_1}\cdot X_2^{S_2}\cdots X_m^{S_m} \le f$. An implicant of f which is maximal is said to be a prime implicant of f. An expression of f which consists of the minimum number of prime implicants is said to be a prime minimal sum-of-products expression. A prime implicant which is contained in every prime minimal sum-of-products expression is said to be an essential prime implicant of f.

Definition 4.3: Let $X_1^{S_1}\cdot X_2^{S_2}\cdots X_m^{S_m}$ be a term. If there are k_j i 's such that $|S_i|=j$ for $j=1,2,\ldots,p$, then this term is said to be a $\underline{k}$-term, where $\underline{k}=(k_1,k_2,\ldots,k_p)$. A $\underline{k}$-term is said to be a $\underline{k}$-cube when it is interpreted geometrically.

Example 4.1: For p=4 and m=4:

$X_1^{\{0,1,3\}}\cdot X_2^{\{0,1\}}\cdot X_3^{\{1,2\}}\cdot X_4^{\{0\}}$ is a (1,2,1,0)-term.

$X_1^{\{0,1,2,3\}}\cdot X_2^{\{1,2\}}\cdot X_3^{\{2,3\}}\cdot X_4^{\{0,1,2\}}$ is a (0,2,1,1)-term.

Definition 4.4: The average number of $\underline{k}$-cubes, prime $\underline{k}$-cubes, and essential $\underline{k}$-cubes of the m-variable functions of weight u are denoted by $G_p(m,\underline{k},u)$, $G_p'(m,\underline{k},u)$, and $G_p''(m,\underline{k},u)$, respectively.

By definition, we have

$$G_p''(m,\underline{k},u) \le G_p'(m,\underline{k},u) \le G_p(m,\underline{k},u)$$

Definition 4.5: The average number of terms in a minimal sum-of-products expression for the functions of m-variable of weight u is denoted by $T_p(m,u)$.

The following lemma is a restatement of (2.1).

Lemma 4.2:

$$\sum_{\underline{k}} G_p''(m,\underline{k},u) \le T_p(m,u) \le \sum_{\underline{k}} G_p'(m,\underline{k},u).$$

By Lemma 4.2, if we can obtain $G_p'(m,\underline{k},u)$ and $G_p''(m,\underline{k},u)$, the upper and lower bounds for $T_p(m,u)$ can be calculated.

Lemma 4.3: Let $N_p(m,\underline{k},u)$, $N_p'(m,\underline{k},u)$, and $N_p''(m,\underline{k},u)$ denote the total number of m-variable functions of weight u which contain a fixed $\underline{k}$-cube, a fixed prime $\underline{k}$-cube, and a fixed essential $\underline{k}$-cubes, respectively. Then

$$G_p(m,\underline{k},u)=\frac{C^{(\underline{k})}}{F^{(u)}}\cdot N_p(m,\underline{k},u);$$

$$G_p'(m,\underline{k},u)=\frac{C^{(\underline{k})}}{F^{(u)}}\cdot N_p'(m,\underline{k},u); \text{ and}$$

$$G_p''(m,\underline{k},u)=\frac{C^{(\underline{k})}}{F^{(u)}}\cdot N_p''(m,\underline{k},u).$$

Where $C^{(\underline{k})}=(m!)\prod_{i=1}^{p}\frac{1}{k_i!}\binom{p}{i}^{k_i}$, $F^{(u)}=\binom{w}{u}$ and $w=p^m$.

Lemma 4.4:

$$N_p(m,\underline{k},u)=\binom{w-w(\underline{k})}{u-w(\underline{k})}, \text{ where } w=p^m \text{ and}$$

$$w(\underline{k})=\prod_{i=1}^{p}(i)^{k_i}.$$

Lemma 4.5:

$$G_p(m,\underline{k},u)=\frac{C^{(\underline{k})}}{F^{(u)}}\cdot\sum_{t=0}^{\eta(\underline{k})}(-1)^t\cdot\sum_{\underline{t}}\lambda(\underline{k},\underline{t})\cdot\binom{w-w(\underline{k},\underline{t})}{u-w(\underline{k},\underline{t})},\qquad\text{------}(4.1)$$

where $F^{(u)}=\binom{w}{u}$, $w=p^m$,

$$C^{(\underline{k})}=(m!)\prod_{i=1}^{p}\left[\frac{1}{k_i!}\binom{p}{i}^{k_i}\right],\quad w(\underline{k},\underline{t})=w(\underline{k})\left\{1+\sum_{i=1}^{p-1}\frac{t_i}{i}\right\},$$

$$w(\underline{k})=\prod_{i=1}^{p}(i)^{k_i},\quad \eta(\underline{k})=\sum_{i=1}^{p-1}a_i,\quad a_i=k_i(p-i),$$

$\underline{t}=(t_1,t_2,\ldots,t_{p-1})$ is a partition of t and

$$\lambda(\underline{k},\underline{t})=\prod_{i=1}^{p-1}\binom{a_i}{i}.$$

When $p=2$, the result of Lemma 4.5 agree with the result of [4].

Lemma 4.6: Let $N_p''(m,\underline{k},u)$ denote the number of functions of weight u such that a fixed $\underline{k}$-cube $C_0^{(\underline{k})}$, and A_j be the corresponding set of $\eta(\underline{k})$ vertices $C_{jh}^{(0)}$ $(h=1,2,\ldots,\eta(\underline{k}))$. Where $C_j^{(0)}\cup C_{jh}^{(0)}$ denotes the $(n-1,1,0,\ldots,0)$-cube not contained in $C_0^{(\underline{k})}$.

Then

$$N_p''(m,\underline{k},u)=\sum_{i=1}^{w(\underline{k})}M(0,j)-\sum_{j_1<j_2}\sum M(0,j_1,j_2)+$$

$$+\sum_{j_1<j_2<j_3}\sum\sum M(0,j_1,j_2,j_3)-\ldots.$$

$$+(-1)^{w(\underline{k})+1}M(0,1,2,\ldots,w(\underline{k})),\text{ where}$$

$$M(0,j_1,j_2,\ldots,j_r)=\binom{w-w(\underline{k})-|A_{j_1}\cup A_{j_2}\cup\ldots\cup A_{j_r}|}{u-w(\underline{k})}$$

Lemma 4.7: For $\underline{k}=(n-1,0,\ldots,0,1,0,\ldots,0)$,

$$N_p''(n,\underline{k},u)=\sum_{i=1}^{w(\underline{k})}(-1)^{i+1}\cdot\binom{w(\underline{k})}{i}\binom{w-w(\underline{k})-\{i(p-1)\cdot(n-1)+(p-s)\}}{u-w(\underline{k})}$$

,where $w(\underline{k})=s$.

Lemma 4.8: For $\underline{k}=(n-2,2,0,\ldots,0)$,

$$N_p''(n,\underline{k},u)=4\binom{w-4-\{(p-1)\cdot(n-2)+2(p-2)\}}{u-4}$$

$$-2\binom{w-4-\{2(p-1)\cdot(n-1)+4(p-2)\}}{u-4}-4\binom{w-4-\{2(p-1)\cdot(n-2)+3(p-2)\}}{u-4}$$

$$+4\binom{w-4-\{3(p-1)\cdot(n-2)+4(p-2)\}}{u-4}-\binom{w-4-\{4(p-1)\cdot(n-2)+4(p-2)\}}{u-4}.$$

Lemma 4.9: For $\underline{k}=(n-k,k)$,

$$N_2''(n,\underline{k},u)=\sum_{i=1}^{w(\underline{k})}(-1)^{i+1}\cdot\binom{w(\underline{k})}{i}\cdot\binom{w-w(\underline{k})-i(n-k)}{u-w(\underline{k})}\quad\text{----}(4.2).$$

The result of Lemma 4.9 agree with the result of [4].

V. Approximation Formulas for the Complexity of Shallow Binary Functions.

In this section, we investigate the properties of shallow binary functions and derive the approximation formulas for the average numbers of prime implicants and essential prime implicants. The values of m and n are supposed to be sufficiently large.

Definition 5.1: Let the weight of the function $f: P^m\to B$ be u, where $P=\{0,1,\ldots,p-1\}$ and $B=\{0,1\}$. The function f is said to be shallow if

$$u\cdot m\cdot\log_2 p\le p^m.$$

Lemma 5.1:

1) If $1\le a<u\le w$, then

$$\left(\frac{u-a}{w-a}\right)^a<\frac{\binom{w-a}{u-a}}{\binom{w}{u}}\le\left(\frac{u}{w}\right)^a.$$

2) If $1\le a\ll u\le w$, then

$$\frac{\binom{w-a}{u-a}}{\binom{w}{u}}\simeq\left(\frac{u}{w}\right)^a.$$

3) If $1\le b\le a\ll u\le w$, then

$$\frac{\binom{w-a}{u-a}}{\binom{w}{u}}\simeq\left(\frac{u}{w}\right)^b\cdot\left(1-\frac{u}{w}\right)^{(a-b)}.$$

Lemma 5.2: $G_p(m,\underline{k},u)\le C^{(\underline{k})}d^{w(\underline{k})}$.

Especially when $w(\underline{k})\ll u$, $G_p(m,\underline{k},u)\simeq C^{(\underline{k})}\cdot d^{w(\underline{k})}$, where $C^{(\underline{k})}=(m!)\prod_{i=1}^{p}\left[\frac{1}{k_i!}\binom{p}{i}^{k_i}\right]$, $w(\underline{k})=\prod_{i=1}^{p}(i)^{k_i}$, and $d=u/p^m$.

Lemma 5.3: If $w(\underline{k})\ll u$, then

$$G_p'(m,\underline{k},u)\simeq C^{(\underline{k})}\cdot d^{w(\underline{k})}\cdot\prod_{i=1}^{p-1}\left(1-d^{w(\underline{k})/i}\right)^{a_i},$$

where $a_i=k_i\cdot(p-i)$ and $d=u/p^m$.

Lemma 5.4: If $\underline{k}=(n-k,k)$ and $w(\underline{k})\ll u$,

$$G_2''(n,\underline{k},u)\simeq\binom{n}{k}\cdot 2^{n-k}\cdot d^{w(\underline{k})}\cdot\{1-[1-(1-d)^{n-k}]^{w(\underline{k})}\},$$

where $d=u/2^n$.

Lemma 5.5: Let $u\cdot m\cdot\log_2 p<p^m$.

1) For $p=2$, except the cubes for $\underline{k}=(m,0)$ and $\underline{k}=(m-1,1)$, $G_p(m,\underline{k},u)$ are small enough to be neglected.
2) For $p=4$, except the cubes for $\underline{k}=(m,0,0,0)$, $\underline{k}=(m-1,1,0,0)$, $\underline{k}=(m-1,0,1,0)$, and $\underline{k}=(m-2,2,0,0)$, $G_p(m,\underline{k},u)$ are small enough to be neglected.

Table 6.1 Average width of two-level PLA's for n-varible functions of density d, and upper and lower bounds on W calculated by (5.1), (4.1), and (4.2).

Two-level PLA

		Average number of essential prime implicants		Average width of PLA's	Average number of prime implicants	
		Exact value	Approximation	Experimental value	Exact value	Approximation
n	d	$\sum G''_2(n,\underline{k},u)$	$A'_0 + A''_1$	$T^*_2(n,u)$	$\sum G'_2(n,\underline{k},u)$	$A'_0 + A'_1$
8	1/32	7.24	7.17	7.2	7.26	7.20
8	2/32	13.17	13.02	13.1	13.49	13.44
8	3/32	18.01	17.69	18.3	19.48	19.38
8	4/32	21.74	21.09	22.5	25.68	25.33
10	1/32	28.09	27.99	29.1	28.31	28.25
10	2/32	50.02	49.68	50.9	53.03	52.87
10	3/32	66.38	65.33	68.4	78.22	77.43
12	1/32	109.53	109.36	109.7	111.29	111.19
12	2/32	190.38	189.20	192.2	210.92	209.96
14	1/32	427.93	427.46	430.7	439.24	438.36

Table 6.2 Average width W of PLA's with two-bit decoders for n-variable functions of density d, and upper and lower bounds on W calculated by (5.2), (5.3), and (4.1)

PLA with two-bit decoders

		Average number of essential prime implicants		Average width of PLA's	Average number of prime implicants		
		Approximation	Approximation	Experimental value	Exact value	Approximation	Approximation
n	d	$B'_0+B''_1+B''_2+B''_3$	$B'_0 + B''_1$	$T^*_4(\frac{n}{2}, u)$	$\sum G'_4(\frac{n}{2},\underline{k},u)$	$B'_0+B'_1+B'_2+B'_3$	$B'_0 + B'_1$
8	1/32	6.82	6.79	6.9	6.96	6.91	6.86
8	2/32	11.89	11.63	12.5	12.75	12.87	12.47
8	3/32	15.45	14.62	16.4	18.56	19.00	17.61
8	4/32	17.54	15.83	19.7	24.86	25.76	22.39
10	1/32	26.38	26.21	28.1	27.05	27.08	26.83
10	2/32	44.31	43.01	46.3	50.92	51.56	49.47
10	3/32	54.52	50.73	61.5	77.25	79.19	71.79
12	1/32	102.03	101.22	102.7	106.44	106.80	105.57
12	2/32	163.97	158.00	173.8	207.09	209.95	199.46
14	1/32	394.45	390.64	403.0	421.89	423.59	417.77

Lemma 5.6: $G_2'(n,\underline{k},u)$ are approximated by the following formulas:

For $\underline{k}=(n,0)$: $\quad 2^n\cdot d\cdot(1-d)^n$;

for $\underline{k}=(n-1,1)$: $n\cdot 2^{n-1}\cdot d^2\cdot(1-d^2)^{n-1}$, where $d=u/2^n$.

Lemma 5.7: $G_2''(n,\underline{k},u)$ are approximated by the following formulas:

For $\underline{k}=(n,0)$ $\quad$: $2^n\cdot d\cdot(1-d)^n$;

for $\underline{k}=(n-1,1)$: $n\cdot 2^{n-1}\cdot d^2\cdot\{1-[1-(1-d)^{n-1}]^2\}$,

where $d=u/2^n$.

Lemma 5.8: $G_4'(m,\underline{k},u)$ are approximated by the following formulas:

For $\underline{k}=(m,0,0,0)$: $4^m\cdot d\cdot(1-d)^{3m}$;

for $\underline{k}=(m-1,1,0,0)$: $\frac{3}{2}\cdot m\cdot 4^m\cdot d^2\cdot(1-d^2)^{3(m-1)}\cdot(1-d)^2$;

for $\underline{k}=(m-1,0,1,0)$: $m\cdot 4^m\cdot d^3\cdot(1-d^3)^{3(m-1)}\cdot(1-d)$; and

for $\underline{k}=(m-2,2,0,0)$: $\frac{9}{8}m(m-1)\cdot 4^m\cdot d^4\cdot(1-d^4)^{3(m-2)}\cdot(1-d^2)^4$,

where $d=u/4^m$.

Lemma 5.9: $G_4''(m,\underline{k},u)$ are approximated by the following formulas:

For $\underline{k}=(m,0,0,0)$: $4^m\cdot d\cdot(1-d)^{3m}$;

for $\underline{k}=(m-1,1,0,0)$: $\frac{3}{2}\cdot m\cdot 4^m\cdot d^2\{2(1-d)^{3m-1}-(1-d)^{6m-4}\}$;

for $\underline{k}=(m-1,0,1,0)$: $m\cdot 4^m\cdot d^3\cdot\{3(1-d)^{3m-2}-(1-d)^{6m-5} + (1-d)^{9m-8}\}$;

for $k=(m-2,2,0,0)$: $\frac{9}{8}m(m-1)4^m\cdot d^4\cdot\{4(1-d)^{3m-2} -2(1-d)^{6m-4}-4(1-d)^{6m-6} +4(1-d)^{9m-10}-(1-d)^{12m-16}\}$,

where $d=u/4^m$.

$T_2(n,u)$ denotes the average width of the AND arrays of two-level PLA's for n-variable functions of weight u, and $T_4(\frac{n}{2},u)$ denotes that of the the PLA's with two-bit decoders. The following theorem is the main result of this paper.

Theorem 5.1: Let $d \le 1/n$ and the value of n be sufficiently large, where $d=u/2^n$.

$$A_0' + A_1'' \le T_2(n,u) \le A_0' + A_1' \qquad ----(5.1).$$

$$B_0'+B_1''+B_2''+B_3'' \le T_4(\tfrac{n}{2},u) \le B_0'+B_1'+B_2'+B_3'' \quad ----(5.2).$$

Where

$A_0'= (1-d)^n u$;

$A_1'=\frac{n}{2}\cdot d\cdot(1-d^2)^{n-1}\cdot u$;

$A_1''=\frac{n}{2}\cdot d\cdot\{1-[1-(1-d)^{n-1}]^2\}u$;

$B_0'= (1-d)^{(3n/2)}\cdot u$;

$B_1'=\frac{3}{4}\cdot d\{(1-d^2)^{(3n/2)-3}\cdot(1-d)^2\}u$;

$B_1''=\frac{3}{4}nd\{2(1-d)^{(3n/2)-1}-(1-d)^{3n-4}\}u$.

$B_2'=\frac{3}{4}\cdot n\cdot d^2\cdot(1-d^3)^{(3n/2)-3}\cdot(1-d)u$;

$B_2''=\frac{1}{2}\cdot nd^2\cdot\{3(1-d)^{(3n/2)-2}-3(1-d)^{3n-5}+(1-d)^{(9n/2)-8}\}u$;

$B_3'=\frac{9}{32}n(n-2)d^3\cdot(1-d^4)^{(3n/2)-6}\cdot(1-d^2)^4 u$;

$B_3''=\frac{9}{32}n(n-2)d^3\cdot\{4(1-d)^{(3n/2)-2}-2(1-d)^{3n-4}-4(1-d)^{3n-6} +4(1-d)^{(9n/2)-10}-(1-d)^{6n-16}\}u$.

Especially when $d \ll 1/n$,

$$B_0' + B_1'' \le T_4(\tfrac{n}{2},u) \le B_0' + B_1' \quad . \qquad ----(5.3)$$

VI. Experimental Results

To investigate the usefulness of the formulas (5.1), (5.2), and (5.3), both expressions of two-valued variables and four-valued variables are minimized for numerous randomly generated functions. Table 6.1 and Table 6.2 show the average width W of PLA's for n-variable functions of density d, and also show the upper and lower bounds on W calculated by using (5.1), (5.2), and (5.3). In the columns labelled "experimental value", each entry denotes the average of ten sample functions. In the case of PLA's with two-bit decoders, assignments of the input variables to the decoders[6] are not optimized. The formulas (5.2) for the bounds on $T_4(\frac{n}{2},u)$ are somewhat complicated. For a rough estimation, we can use the bounds (5.3) instead of (5.2). These tables also show exact values of average numbers of prime implicants and essential prime implicants which are calculated by using (4.1) and (4.2). From these tables, we can conclude that the formulas are useful for estimation of the average size of PLA's.

VII. Conclusion and Comments

In this paper, several properties of shallow binary functions are obtained, and approximation formulas for the average numbers of prime implicants and essential prime implicants are derived. By using these formulas, we can estimate the average size of PLA's for shallow logic functions.

It is known that the width of PLA's with two-bit decoders can be reduced by optimizing the assignment of the input variables[6]. For example, our experiment shows that optimally assigned PLA's are 7.2% smaller than non-optimally ones in the case of n=12 and d=1/32. The formulas for the optimally assigned PLA's remain to be derived.

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