IEICE TRANS. FUNDAMENTALS, VOL.E87–A, NO.5 MAY 2004

PAPER Special Section on Discrete Mathematics and Its Applications

# Area-Time Complexities of Multi-Valued Decision Diagrams

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**SUMMARY** This paper considers Quasi-Reduced ordered Multi-valued Decision Diagrams with $k$ bits (QRMDD($k$)s) to represent binary logic functions. Experimental results show relations between the values of $k$ and the numbers of nodes, the memory sizes, the numbers of memory accesses, and area-time complexity for QRMDD($k$). For many benchmark functions, the numbers of nodes and memory accesses for QRMDD($k$)s are nearly equal to $\frac{1}{k}$ of the corresponding Quasi-Reduced ordered Binary Decision Diagrams (QRBDDs), and the memory sizes and the area-time complexities for QRMDD($k$)s are minimum when $k = 2$ and $k = 3$–6, respectively.

***key words:*** *decision diagrams, the number of nodes, area-time complexity, randomly generated function, representation of logic functions*

## 1. Introduction

Binary Decision Diagrams (BDDs) [7] and Multi-valued Decision Diagrams (MDDs) [3], [14], [19], [20] are extensively used in logic synthesis [9], logic simulation [1], [11], [17], software synthesis [2], etc. Since modern computer systems have the memory hierarchical structure, suitable decision diagrams for the memory hierarchy can shorten the runtimes of these applications [17], [34]. Quasi-Reduced ordered BDDs (QRBDDs) and Quasi-Reduced ordered MDDs (QRMDDs) are suitable for the memory hierarchy [23], parallel process [12], [24], and design of LUT cascades [31]. However, in general, QRBDDs and QRMDDs require more nodes than corresponding Reduced ordered BDDs (RBDDs) and Reduced ordered MDDs (RMDDs) to represent logic functions. Hence, the minimizations of QRBDDs and QRMDDs are very important. In many cases, the minimizations of decision diagrams use the variable reordering [8], [9], [13], [21], [26]. In the minimization of MDDs, a partition of binary variables [10], [28] is important as well as the variable ordering.

To represent a binary logic function using an MDD, binary variables are partitioned into groups. The papers [10], [28] present the optimization algorithm of partition of input binary variables into groups of binary variables. However, the size of groups (i.e. the number of binary variables in a group) is fixed in these algorithms. In this paper, we assume that the size of groups, that is the value of $k$ for Quasi-Reduced ordered MDDs with $k$ bits (QRMDD($k$)s), can be changed, and we find the optimum sizes of groups experimentally by showing the relations of the values of $k$ and the numbers of nodes, the memory sizes, and the numbers of memory accesses. To show these relations, we assume that the order of binary variable is fixed. Our statistical results are useful for minimizations of MDDs, software synthesis [2], and logic simulation [1], [11], [17].

The rest of the paper is organized as follows: Sect. 2 defines MDD($k$)s, QRMDD($k$)s, computation model for MDDs, and a method to represent multiple-output functions. Section 3 considers the number of nodes in QRMDD($k$)s for general functions, benchmark functions, and randomly generated functions. Section 4 introduces the measure called *area-time complexity* [4], [33] to find the optimum value of $k$ for QRMDD($k$)s, and derives the optimum values of $k$ by experiments.

This paper is an extended version of [22].

## 2. Definitions

### 2.1 Partitions of Binary Variables

**Definition 2.1:** Let $f(X)$ be a two-valued logic function, where $X = (x_1, x_2, \ldots, x_n)$, and $x_i$ $(i = 1, 2, \ldots, n)$ are binary variables. And, let $\{X\}$ denote the set of variables in $X$. If $\{X\} = \{X_1\} \cup \{X_2\} \cup \ldots \cup \{X_u\}$ and $\{X_i\} \cap \{X_j\} = \phi$ $(i \neq j)$, then $(X_1, X_2, \ldots, X_u)$ is a **partition** of $X$. An ordered set of variables $X_i$ is called a **super variable**. If $|X_i| = k$ $(i = 1, 2, \ldots, u)$, then a two-valued logic function $f(X)$ can be represented by the mapping $f(X_1, X_2, \ldots, X_u)$: $\{0, 1, 2, \ldots, 2^k - 1\}^u \rightarrow \{0, 1\}$.

### 2.2 QRMDD($k$)

In this paper, we use the standard terminologies for Reduced ordered Binary Decision Diagrams (RBDDs) [7] and Reduced ordered MDDs (RMDDs) [14].

**Definition 2.2:** When $X = (x_1, x_2, \ldots, x_n)$ is partitioned

Manuscript received August 22, 2003.
Manuscript revised November 10, 2003.
Final manuscript received December 26, 2003.

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into $(X_1, X_2, \ldots, X_u)$, where $|X_i| = k$ $(i = 1, 2, \ldots, u)$, an RMDD representing a logic function $f(X)$ is called a **Reduced ordered MDD with $k$ bits (RMDD($k$))**. In an RMDD($k$), non-terminal nodes have $2^k$ outgoing edges. When $k = 1$, an RMDD(1) is an RBDD.

If $|X_1| = |X_2| = \ldots = |X_{u-1}| = k$ and $|X_u| < k$, then for $|X_u| = k$, we use redundant binary variables which are called **dummy variables**. The set of binary variables with dummy variables is denoted by $\{X'\} = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+t}\}$, where $|X'| = n + t$, and $t$ denotes the number of dummy variables. Note that $f$ is independent of $x_{n+1}, x_{n+2}, \ldots$ and $x_{n+t}$.

**Definition 2.3:** In a decision diagram (DD), a path from the root node to a terminal node is a **path of DD**. The number of non-terminal nodes on the path is the **length of the path**.

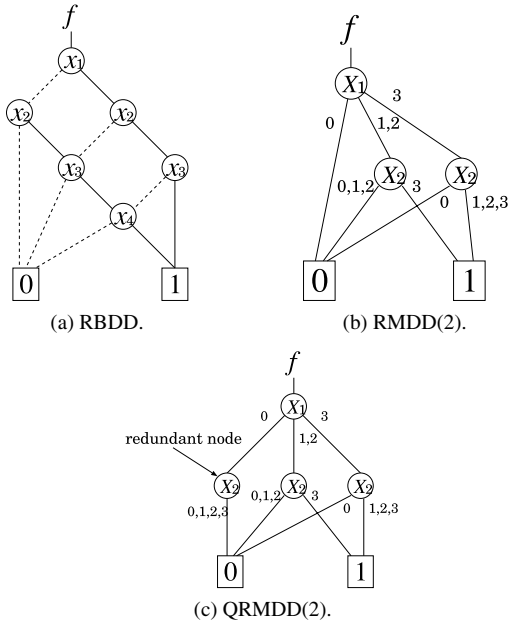
**Definition 2.4:** In a DD, the **number of nodes in the DD**, denoted by *nodes*(DD), includes only non-terminal nodes.

**Definition 2.5:** When all $X_i$ $(i = 1, 2, \ldots, u)$ appear in this order on an arbitrary path of an MDD($k$), the MDD($k$) is a **Quasi-Reduced ordered MDD with $k$ bits (QRMDD($k$))**. QRMDD($k$) has no isomorphic subgraphs.

The length of an arbitrary path in a QRMDD($k$) is the number of super variables. An RMDD($k$) has no redundant nodes, while a QRMDD($k$) usually has redundant nodes. Therefore, we have the following relation in the number of nodes of an RMDD($k$) and its corresponding QRMDD($k$):

$$nodes(\text{RMDD}(k)) \leq nodes(\text{QRMDD}(k)).$$

**Example 2.1:** Consider a logic function $f = x_1x_2x_3 \vee x_2x_3x_4 \vee x_3x_4x_1 \vee x_4x_1x_2$. The RBDD, the RMDD(2), and the QRMDD(2) for $f$ are shown in Figs. 1(a), (b), and (c), respectively. In Fig. 1(a), the solid lines and the broken lines denote 1-edges and 0-edges, respectively. In Figs. 1(b) and (c), the input variables $X = (x_1, x_2, x_3, x_4)$ are partitioned into $(X_1, X_2)$, where $X_1 = (x_1, x_2)$ and $X_2 = (x_3, x_4)$. We have *nodes*(RBDD) = 6, *nodes*(RMDD(2)) = 3, and *nodes*(QRMDD(2)) = 4. (End of Example)




(a) RBDD. (b) RMDD(2).

(c) QRMDD(2).

**Fig. 1** DDs for $f$.

### 2.3 Computation Model for MDDs

In an MDD, we assume the following computation model:

1. MDD($k$)s for logic functions are evaluated by traversing nodes from the root node to a terminal node according to values of the super variables.
2. MDDs are implemented directly, not simulated using the BDD package as described in [14].
3. Encoded input values are available, and their access time is negligible. For example, when $X_1 = (x_1, x_2, x_3, x_4) = (1, 0, 0, 1)$, $X_1 = 9$ is available as an input to the super variable.
4. Most computation time is spent for accessing nodes.
5. The access time to all MDD nodes are equal.

In this case, the time to evaluate a QRMDD for a logic function is proportional to the path length (i.e., the number of super variables).

### 2.4 Representations of Multiple-Output Functions

Logic networks usually have many outputs. In most cases, independent representation of each output is inefficient. Let the multiple-output functions be $F = (f_0, f_1, \ldots, f_{m-1})$: $B^n \rightarrow B^m$, where $B = \{1, 0\}$, and $n$ and $m$ denote the number of input variables and outputs, respectively. Several methods exist to represent multiple-output functions by using DDs [18], [27]–[29], [32]. In this paper, we use an encoded characteristic function for non-zero output (ECFN) [30], [32] to represent multiple-output functions. An ECFN uses $\lceil \log_2 m \rceil$ auxiliary variables to represent the outputs. In the following, a DD means a DD for an ECFN.

**Definition 2.6:** The **density** for an $n$-variable logic function $f$ is defined as

$$\frac{|f|}{2^n} \times 100,$$

where $|f|$ denotes the number of $\vec{a}$ such that $f(\vec{a}) = 1$.

The density for a multiple-output function $F$ is the density for an ECFN representing $F$.

## 3. Number of Nodes in QRMDD($k$)

In this section, we first obtain an upper bound on the number of nodes in a QRMDD($k$). Then, we obtain the numbers of nodes in QRMDD($k$)s for benchmark functions, and show that an interesting property holds for many benchmark functions. Finally, we obtain the numbers of nodes in

**Table 1** Upper bounds on the number of nodes in QRMDD($k$).

| | $k$ | | | | |
|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 |
| 10 | 275 | 101 | 77 | 33 | 33 |
| 11 | 531 | 345 | 89 | 273 | 37 |
| 12 | 787 | 357 | 329 | 273 | 49 |
| 13 | 1299 | 601 | 589 | 277 | 289 |
| 14 | 2323 | 1381 | 601 | 289 | 1057 |
| 15 | 4371 | 1625 | 841 | 529 | 1057 |
| 16 | 8467 | 5477 | 4685 | 4369 | 1061 |
| 17 | 16659 | 5721 | 4697 | 4373 | 1073 |
| 18 | 33043 | 21861 | 4937 | 4385 | 1313 |
| 19 | 65811 | 22105 | 37453 | 4625 | 33825 |
| 20 | 131347 | 87397 | 37465 | 69905 | 33825 |

QRMDD($k$)s for randomly generated functions, and show that they have quite different property from the benchmark functions.

### 3.1 Number of Nodes for General Functions

**Theorem 3.1:** An arbitrary $n$-variable logic function can be represented by a QRBDD with at most

$$2^{n-r} - 1 + \sum_{i=1}^{r} 2^{2^i}$$

non-terminal nodes, where $r$ is the largest integer that satisfies relation $n - r \geq 2^r$ [16].

**Theorem 3.2:** An arbitrary $n$-variable logic function can be represented by a QRMDD($k$) with at most

$$\frac{2^{sk} - 1}{2^k - 1} + \sum_{i=1}^{u-s} 2^{2^{(ki-t)}}$$

non-terminal nodes, where $u$ is the number of super variables, $t$ is the number of dummy variables, and $s$ is the smallest integer that satisfy relation

$$s \geq \frac{n-r}{k}.$$

Appendix B and Appendix C show the proofs of Theorem 3.1 and Theorem 3.2.

Table 1 shows the upper bounds on the number of nodes in QRMDD($k$)s for $n$-variable logic functions. We can see that the upper bounds are non-monotone functions of $k$.

### 3.2 Number of Nodes for Benchmark Functions

We used 157 benchmark functions [5], [29], [36] in Table A· 1. In this paper, encodings for ECFNs and binary variable orders of BDDs are obtained by the heuristic algorithm in [32]. In the following experiments, we use these variable orders, and we consider only the partition of binary variables. For each benchmark function, we counted the number of nodes in the corresponding QRMDD($k$)s for various $k$. In Table 2, *avg* denotes the arithmetic average of the relative numbers of nodes, where the number of nodes in QRBDD is set to 1.00, and *stdv* denotes the standard deviation.

**Table 2** Relation of nodes in QRMDD($k$) and $k$ for benchmark functions.

| | $k$ | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| *avg* | 1.000 | 0.498 | 0.333 | 0.248 | 0.202 |
| *stdv* | 0.000 | 0.013 | 0.009 | 0.016 | 0.016 |

**Table 3** Benchmark functions with $\eta \geq 0.1$.

| Name | # in | # out | # nodes | dens. | cate. |
|---|---|---|---|---|---|
| C499 | 41 | 32 | 24476 | 50.0 | 1 |
| C1355 | 41 | 32 | 30156 | 50.0 | 1 |
| C1908 | 33 | 25 | 9292 | 45.8 | 1 |
| adr8 | 16 | 9 | 153 | 50.0 | 2 |
| adr9 | 18 | 10 | 180 | 50.0 | 2 |
| comp | 32 | 3 | 114 | 37.5 | 2 |
| inc17 | 17 | 18 | 236 | 48.4 | 3 |
| log16 | 16 | 16 | 11216 | 59.9 | 1 |
| log17 | 17 | 17 | 23054 | 55.1 | 1 |
| log18 | 18 | 18 | 31458 | 55.2 | 1 |
| mlp8 | 16 | 16 | 10112 | 41.5 | 1 |
| mlp9 | 18 | 18 | 28332 | 37.5 | 1 |
| mlp10 | 20 | 20 | 82077 | 38.5 | 1 |
| my_adder | 33 | 17 | 450 | 50.0 | 2 |
| nrm8 | 16 | 9 | 8689 | 49.1 | 1 |
| nrm9 | 18 | 10 | 23152 | 49.0 | 1 |
| pcle | 19 | 9 | 221 | 29.3 | 3 |
| rot16 | 16 | 9 | 1021 | 60.8 | 3 |
| rot17 | 17 | 9 | 1429 | 49.3 | 3 |
| sqr16 | 16 | 32 | 18366 | 42.9 | 1 |
| tcon | 17 | 16 | 183 | 50.0 | 3 |
| vg2 | 25 | 8 | 217 | 22.9 | 3 |
| vtx1 | 27 | 6 | 326 | 12.5 | 3 |
| x1dn | 27 | 6 | 332 | 12.5 | 3 |

**Definition 3.1:** The relation '$\simeq$' is defined as follows:

$$a \simeq b \Leftrightarrow \eta < 0.1,$$

where $a$ and $b$ are positive integers, and the **normalized difference** $\eta$ is given by:

$$\eta = \frac{|a-b|}{\min(a,b)}.$$

If $a \simeq b$, then $a$ and $b$ are **nearly equal**.

For 133 functions in Table A· 1, the following property holds.

**Property 3.1:**

$$nodes(\text{QRMDD}(k)) \simeq \frac{1}{k} nodes(\text{QRBDD})$$

For the remaining 24 functions, $\eta \geq 0.1$ holds. Table 3 lists these 24 functions. In Table 3, "# nodes," "dens.," and "cate." denote the numbers of nodes in QRBDDs, the densities, and the categories of functions described below, respectively. Figure 2 shows the relation between the normalized difference $\eta$ and the density for benchmark functions. The symbols +,×,⊙, and △ correspond to the values for $k = 2, 3, 4$, and 5, respectively. For each function, we assume that Property 3.1 holds when all the symbols are below the border line of $\eta = 0.1$ (i.e., $\eta < 0.1$ holds for $k =$ 2–5). From Fig. 2, we categorized 24 functions in Table 3 into three sets.

1. The densities of functions are between 40% and 60%, and the number of nodes for QRBDDs are large relative to the number of inputs.

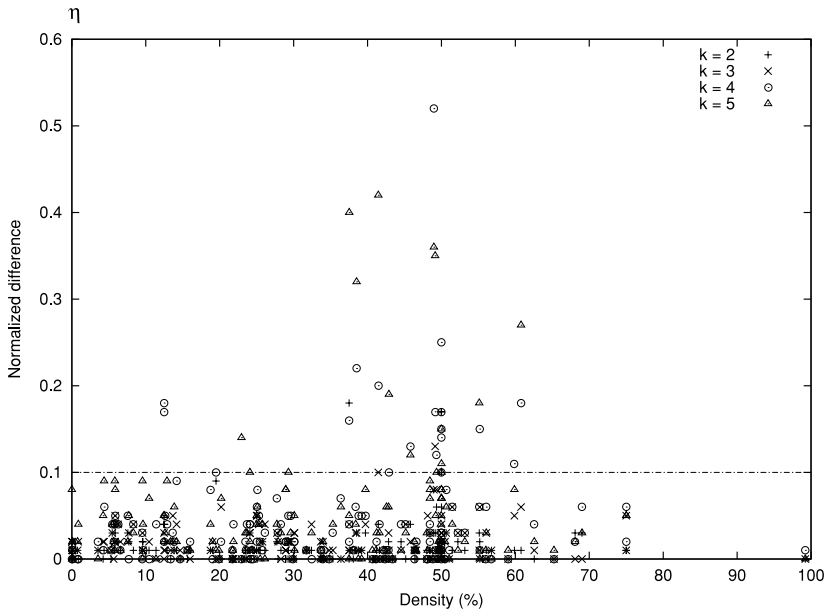



**Fig. 2** Relation between the normalized difference $\eta$ and density for benchmark functions.

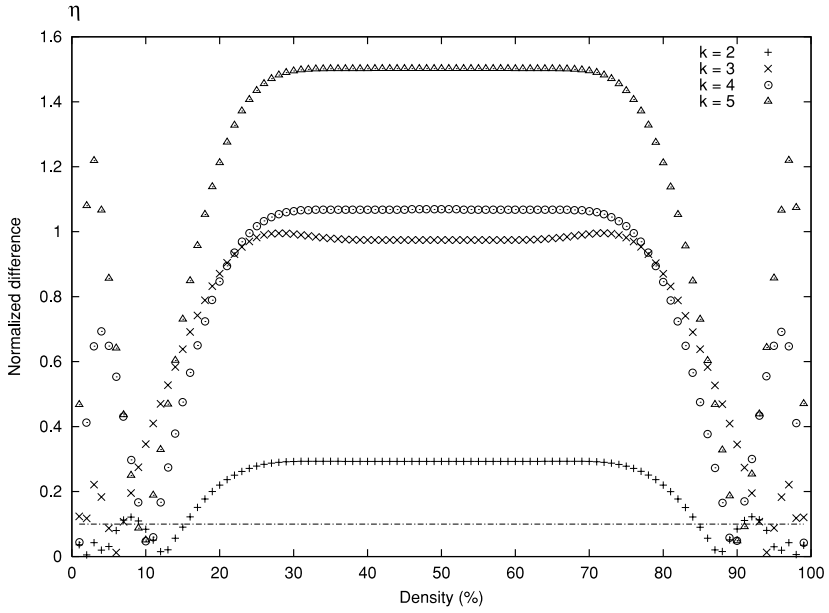



**Fig. 3** Relation between the normalized difference $\eta$ and density for randomly generated functions.

2. The functions have iterative properties (i.e., adder and comparator).
3. The numbers of nodes, inputs, and outputs are small. Property 3.1 does not hold for $k = 4$ or 5.

### 3.3 Number of Nodes for Randomly Generated Functions

For $d = 1, 2, \ldots, 99$, we randomly generated one 25-variable function with density $d$ to obtain 99 functions. Figure 3 shows the relation between the normalized difference $\eta$ and the density for randomly generated functions. In this case, no randomly generated functions of 25 variables satisfied Property 3.1. This fact shows that randomly generated functions have quite different property from the benchmark functions in Table A·1. For many benchmark functions, the numbers of nodes in QRMDD($k$)s decrease as $k$ increase. However, for randomly generated functions, the number of nodes is a non-monotone function of $k$. For example, for many randomly generated functions of 25 variables, the numbers of nodes in QRMDD(5)s were larger than those in QRMDD(3)s.

For $n = 10, 11, \ldots, 20$, we also randomly generated ten $n$-variable functions with density 50%. Table 4 shows the average numbers of nodes in QRMDD($k$)s for randomly

**Table 4** Number of nodes in QRMDD($k$) for randomly generated functions.

| | $k$ | | | | |
|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 |
| 10 | 247.4 | 101.0 | 77.0 | 33.0 | 33.0 |
| 11 | 437.1 | 251.2 | 89.0 | 179.2 | 37.0 |
| 12 | 754.0 | 356.5 | 296.5 | 272.5 | 49.0 |
| 13 | 1292.8 | 596.6 | 587.2 | 277.0 | 284.6 |
| 14 | 2316.0 | 1374.1 | 601.0 | 289.0 | 1050.1 |
| 15 | 4341.1 | 1625.0 | 841.0 | 529.0 | 1057.0 |
| 16 | 8336.5 | 5346.5 | 4554.5 | 4238.5 | 1061.0 |
| 17 | 16165.3 | 5721.0 | 4697.0 | 4373.0 | 1073.0 |
| 18 | 31155.9 | 19973.9 | 4937.0 | 4385.0 | 1313.0 |
| 19 | 58836.4 | 22105.0 | 30478.4 | 4625.0 | 26850.4 |
| 20 | 107220.3 | 63270.3 | 37465.0 | 45778.3 | 33825.0 |

generated functions. The deviations were within ±2% of the averages. From Table 1 and Table 4, we can see that the numbers of nodes in QRMDD($k$)s for randomly generated functions with density 50% are nearly equal to the upper bounds.

## 4. Area-Time Complexity of QRMDD($k$)

### 4.1 Memory Size for QRMDD($k$)

**Definition 4.1:** The **memory size for a QRMDD($k$)** is the number of bits needed to store the QRMDD($k$) in memory.

In memory, a non-terminal node in an RMDD($k$) requires an index and a set of pointers that refer the succeeding nodes. However, in a QRMDD($k$), each non-terminal node has no index because $X_1, X_2, \ldots, X_u$ are evaluated always in this order, and the index of the super variable to evaluate can be obtained by a counter, where the super variable order is $X_1, X_2, \ldots, X_u$.

**Example 4.1:** Figure 4 illustrates data structures of a node in an RMDD(2) and a QRMDD(2). (End of Example)

Because each node in a QRMDD($k$) has $2^k$ outgoing edges, we need

$$2^k nodes(\text{QRMDD}(k))$$

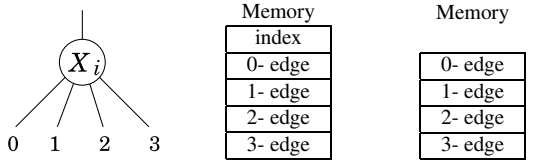
words to store all nodes in a QRMDD($k$). Since each node in a memory requires a unique address, each pointer requires

$$\lceil \log_2(nodes(\text{QRMDD}(k))) \rceil$$

bits to specify the address. Therefore, the memory size for a QRMDD($k$) is

$$2^k nodes(\text{QRMDD}(k)) \lceil \log_2(nodes(\text{QRMDD}(k))) \rceil.$$

As shown in Sect. 3.2, for many benchmark functions, the numbers of nodes in QRMDD($k$)s can be reduced with increasing $k$. On the other hand, the memory sizes for QRMDD($k$)s increase with $2^k$. This fact shows that in QRMDD($k$), there exists optimum value of $k$ that minimizes the memory size.




(a) A node in DDs. (b) RMDD(2). (c) QRMDD(2).

**Fig. 4** Data structure of a node in DDs.

### 4.2 Area-Time Complexity of QRMDD($k$)s

Because a QRMDD($k$) evaluates $k$ binary variables at a time, the number of memory accesses of a QRMDD($k$) is $\frac{1}{k}$ of the corresponding QRBDD. On the other hand, the memory size for a QRMDD($k$) increases with $2^k$. In this section, we consider the area-time complexity [4], [33] for QRMDD($k$) and obtain the $k$ that minimizes the area-time complexity.

**Definition 4.2:** The **area-time complexity** is the measure of computational cost considering both area and time. It is defined by

$$AT = (area) \times (time), \quad AT^2 = (area) \times (time)^2.$$

In this paper, the area $A$ corresponds to the necessary memory size for QRMDD($k$), and the time $T$ corresponds to the number of memory accesses to evaluate logic function.

The measure $AT$ is used when both the memory size and the number of memory accesses are equally important. The measure $AT^2$ is used when the number of memory accesses is more important than the memory size. For example, $AT$ can be used for software synthesis, while $AT^2$ can be used for logic simulators. In the software synthesis for embedded systems [2], compact and fast program codes are required because of the memory limitations and the time limitations for systems. Thus, in the software synthesis using DDs, the optimization of DDs considering both the memory size and the number of memory accesses is important. In logic simulators [1], [11], [17], fast evaluation of logic functions is more important to reduce the design verification time. Thus, in logic simulators, minimizing the number of memory accesses using a reasonable amount of memory is important.

### 4.3 Experimental Results

For each benchmark function in Table A·1, we obtained three measures $A$, $AT$, and $AT^2$. Table 5, Table 6, and Table 7 show the relations of $k$ and $A$, $AT$, and $AT^2$, respectively. In these tables, *avg* denotes the arithmetic average, and *stdv* denotes the standard deviation for benchmark functions.

For each benchmark function in Table A·1, $A$ takes its minimum when $k = 2$; $AT$ takes its minimum when $k = 3$ or $k = 4$; and $AT^2$ takes its minimum when $k = 4$–6.

**Table 5** Relation of $k$ and $A$ for QRMDD($k$) for benchmark functions.

| | $k$ | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| *avg* | 1.00 | 0.90 | 1.14 | 1.61 | 2.54 |
| *stdv* | 0.000 | 0.035 | 0.079 | 0.144 | 0.292 |

**Table 6** Relation of $k$ and $AT$ for QRMDD($k$) for benchmark functions.

| | $k$ | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| *avg* | 1.00 | 0.46 | 0.39 | 0.42 | 0.54 |
| *stdv* | 0.000 | 0.019 | 0.030 | 0.039 | 0.070 |

**Table 7** Relation of $k$ and $AT^2$ for QRMDD($k$) for benchmark functions.

| | $k$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| *avg* | 1.000 | 0.232 | 0.133 | 0.110 | 0.114 | 0.128 | 0.167 |
| *stdv* | 0.000 | 0.011 | 0.012 | 0.012 | 0.019 | 0.023 | 0.046 |

### 4.4 Analysis for the Functions that Satisfy Property 3.1

In Sect. 4.3, for QRMDD($k$)s, we found the values of $k$ that make $A$, $AT$, and $AT^2$ minimum, experimentally. In this section, we assume that Property 3.1 holds, and will find the values $k$ that make $A$, $AT$, and $AT^2$ minimum, analytically. Let $A$ and $T$ be the memory size for a QRMDD($k$) and the number of memory accesses necessary to evaluate a QRMDD($k$), respectively. Then, we have the following:

$$A = 2^k nodes(\text{QRMDD}(k))\lceil \log_2(nodes(\text{QRMDD}(k)))\rceil,$$
$$T = \left\lceil \frac{n}{k} \right\rceil.$$

Let $nodes(\text{QRMDD}(1)) = N$ and assume that Property 3.1 holds. Then we have:

$$A \simeq \frac{2^k}{k} N \left\lceil \log_2\left(\frac{N}{k}\right)\right\rceil,$$
$$AT \simeq \frac{2^k n}{k^2} N \left\lceil \log_2\left(\frac{N}{k}\right)\right\rceil,$$
$$AT^2 \simeq \frac{2^k n^2}{k^3} N \left\lceil \log_2\left(\frac{N}{k}\right)\right\rceil.$$

Note that $N$ is usually so greater than 200, while $k$ is usually at most 7. Thus, we can use the following approximation:

$$\lceil \log_2(N) - \log_2(k)\rceil \simeq \lceil \log_2(N)\rceil.$$

Therefore, $A$, $AT$, and $AT^2$ can be simplified to

$$A \simeq \frac{2^k}{k} C_0, \quad AT \simeq \frac{2^k}{k^2} C_1, \quad \text{and} \quad AT^2 \simeq \frac{2^k}{k^3} C_2,$$

respectively, where the constants $C_0$, $C_1$ and $C_2$ are independent of $k$. From the above formulas, we can see that $A$, $AT$, and $AT^2$ take their minimum when $k = 2$, $k = 3$, and $k = 4$, respectively.

## 5. Conclusion and Comments

In this paper, we considered representations of logic functions using QRMDD($k$)s. Experimental results showed that: 1) For many benchmark functions, the numbers of nodes in QRMDD($k$)s are nearly equal to $\frac{1}{k}$ of the corresponding QRBDDs. On the other hand, for randomly generated functions, the number of nodes is a non-monotone function of $k$. 2) For many benchmark functions, the memory sizes and the area-time complexities for QRMDD($k$)s take their minimum when $k = 2$ and $k = 3$–6, respectively.

In commercial LUT-based FPGAs, the numbers of inputs $k$ for LUT cells are usually between 4 and 6 [6]. The studies in [15], [25] show that when $k = 4$–6, the architectures of FPGAs are optimum. The cost of $k$-LUT cell increases with $k$, while the level of network reduces with $k$. Thus, in logic synthesis with FPGAs, we can do a similar discussion. However, the optimum value of $k$ for FPGAs depends on interconnection delay, logic synthesis tools, and process technology as well as the cost of $k$-LUT cell and the level of networks [35]. It is interesting that in both cases, the optimum values of $k$ are 4–6 even if they have different cost functions.

## Acknowledgments

This research is partly supported by the Grant in Aid for Scientific Research of The Japan Society for the Promotion of Science (JSPS), and the Takeda Foundation. Prof. Jon. T. Butler improved presentation. The comments of a reviewer motivated to produce Fig. 2 and Fig. 3.

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## Appendix A: Benchmark Functions

In this paper, we use 157 benchmark functions [5], [29], [36] shown in Table A· 1, where $n$ and $m$ denote the number of input and output variables, respectively. In this table, the benchmark functions under *sequential* originally represented sequential circuits. We removed flip-flops (FFs) from these sequential circuits to make them combinational. Such functions are renamed by appending a subscript '$c$' to the original names.

## Appendix B: Proof of Theorem 3.1

**Definition B.1:** Suppose that a QRBDD for an $n$-variable logic function is partitioned into two parts as shown in Fig. A· 1. It is partitioned into **the upper part** which has the variables $X_1=(x_1, x_2, \ldots, x_{n-r})$, and **the lower part** which has the variables $X_2=(x_{n-r+1}, \ldots, x_n)$. In this case, the BDD represents the logic function as follows:

$$f(X_1, X_2) = \bigvee_{\vec{a}_i \in B^{n-r}} X_1^{\vec{a}_i} f(\vec{a}_i, X_2),$$

where

$$X_1^{\vec{a}_i} = \begin{cases} 1 & (X_1 = \vec{a}_i) \\ 0 & (otherwise). \end{cases}$$

The upper part realizes $X_1^{\vec{a}_i}$, and the lower part realizes $f(\vec{a}_i, X_2)$.

**(Proof of Theorem 3.1)** When the upper part of the QRBDD (see Fig. A· 1) has $2^{n-r} - 1$ nodes (i.e., a complete binary tree), it is the maximum. Because $f(\vec{a}_i, X_2)$ is an $r$-variable logic function, the number of different $f(\vec{a}_i, X_2)$ is $2^{2^r}$. When $2^{2^i}$ logic functions are realized for each level $i$ ($i = 1, 2, \ldots, r$) from the terminal node to the $r$, the lower part is the maximum. Therefore, the number of nodes in a QRBDD is at most

$$2^{n-r} - 1 + \sum_{i=1}^{r} 2^{2^i}.$$

**Table A· 1** Benchmark functions.

| Name | n | m | Name | n | m | Name | n | m |
|---|---|---|---|---|---|---|---|---|
| 3adr6 | 18 | 12 | i5 | 133 | 66 | signet | 39 | 8 |
| C432 | 36 | 7 | i6 | 138 | 67 | soar | 83 | 94 |
| C499 | 41 | 32 | i7 | 199 | 67 | spla | 16 | 46 |
| C880 | 60 | 26 | i8 | 133 | 81 | sqr16 | 16 | 32 |
| C1355 | 41 | 32 | i9 | 88 | 63 | t1 | 21 | 23 |
| C1908 | 33 | 25 | i10 | 257 | 224 | t2 | 17 | 16 |
| C2670 | 233 | 140 | ibm | 48 | 17 | table5 | 17 | 15 |
| C3540 | 50 | 22 | in1 | 16 | 17 | tcon | 17 | 16 |
| C5315 | 178 | 123 | in2 | 19 | 10 | term1 | 34 | 10 |
| C7552 | 207 | 108 | in3 | 35 | 29 | ti | 47 | 72 |
| accpla | 50 | 69 | in4 | 32 | 20 | too_large | 38 | 3 |
| adr8 | 16 | 9 | in5 | 24 | 14 | ts10 | 22 | 16 |
| adr9 | 18 | 10 | in6 | 33 | 23 | ttt2 | 24 | 21 |
| al2 | 16 | 47 | in7 | 26 | 10 | unreg | 36 | 16 |
| alcom | 15 | 38 | inc16 | 16 | 17 | vda | 17 | 39 |
| apex1 | 45 | 45 | inc17 | 17 | 18 | vg2 | 25 | 8 |
| apex2 | 39 | 3 | inc18 | 18 | 19 | vtx1 | 27 | 6 |
| apex3 | 54 | 50 | jbp | 36 | 57 | wgt17 | 17 | 5 |
| apex5 | 117 | 88 | k2 | 45 | 45 | wgt18 | 18 | 5 |
| apex6 | 135 | 99 | lal | 26 | 19 | x1 | 51 | 35 |
| apex7 | 49 | 37 | log16 | 16 | 16 | x3 | 135 | 99 |
| b2 | 16 | 17 | log17 | 17 | 17 | x4 | 94 | 71 |
| b3 | 32 | 20 | log18 | 18 | 18 | x1dn | 27 | 6 |
| b4 | 33 | 23 | mainpla | 27 | 54 | x2dn | 82 | 56 |
| b9 | 41 | 21 | mark1 | 20 | 31 | x6dn | 39 | 5 |
| bc0 | 26 | 11 | misex2 | 25 | 18 | x7dn | 66 | 15 |
| bca | 26 | 46 | misg | 56 | 23 | x9dn | 27 | 7 |
| bcb | 26 | 39 | mish | 94 | 43 | xparc | 41 | 73 |
| bcc | 26 | 45 | misj | 35 | 14 | sequential | | |
| bcd | 26 | 38 | mlp8 | 16 | 16 | $s208_c$ | 18 | 9 |
| c8 | 28 | 18 | mlp9 | 18 | 18 | $s298_c$ | 17 | 20 |
| cc | 21 | 20 | mlp10 | 20 | 20 | $s344_c$ | 24 | 26 |
| chkn | 29 | 7 | mux | 21 | 1 | $s349_c$ | 24 | 26 |
| cht | 47 | 36 | my_adder | 33 | 17 | $s382_c$ | 24 | 27 |
| cm150a | 21 | 1 | nrm8 | 16 | 9 | $s400_c$ | 24 | 27 |
| comp | 32 | 3 | nrm9 | 18 | 10 | $s420_c$ | 34 | 17 |
| cordic | 23 | 2 | opa | 17 | 69 | $s444_c$ | 24 | 27 |
| count | 35 | 16 | pair | 173 | 137 | $s510_c$ | 25 | 13 |
| cps | 24 | 109 | pcle | 19 | 9 | $s526_c$ | 24 | 27 |
| dalu | 75 | 16 | pcler8 | 27 | 17 | $s641_c$ | 54 | 43 |
| des | 256 | 245 | pdc | 16 | 40 | $s713_c$ | 54 | 42 |
| dk48 | 15 | 17 | pm1 | 16 | 13 | $s820_c$ | 23 | 24 |
| duke2 | 22 | 29 | rckl | 32 | 7 | $s832_c$ | 23 | 24 |
| e64 | 65 | 65 | rdm16 | 16 | 16 | $s838_c$ | 66 | 33 |
| ex4 | 128 | 28 | rdm17 | 17 | 17 | $s1196_c$ | 32 | 32 |
| example2 | 85 | 66 | rdm18 | 18 | 18 | $s1423_c$ | 91 | 79 |
| exep | 30 | 63 | rot | 135 | 107 | $s5378_c$ | 214 | 228 |
| frg1 | 28 | 3 | rot16 | 16 | 9 | $s9234_c$ | 247 | 250 |
| frg2 | 143 | 139 | rot17 | 17 | 9 | $s13207_c$ | 700 | 790 |
| i1 | 25 | 16 | rot18 | 18 | 10 | $s15850_c$ | 611 | 684 |
| i2 | 201 | 1 | sct | 19 | 15 | $s38417_c$ | 1664 | 1742 |
| i3 | 132 | 6 | seq | 41 | 35 | $s38584_c$ | 1464 | 1730 |
| i4 | 192 | 6 | shift | 19 | 16 | | | |

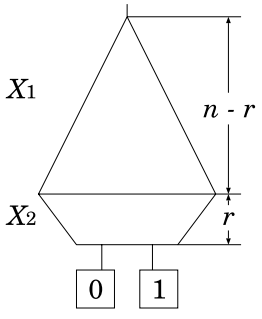


**Fig. A· 1** Partition of QRBDD.

This upper bound becomes the tightest when $r$ is the maximum integer satisfying $n - r \geq 2^r$ [16]. (Q.E.D.)

## Appendix C: Proof of Theorem 3.2

The proof of Theorem 3.2 also uses similar approach.

**(Proof of Theorem 3.2)** Since each node in a QRMDD($k$) has $2^k$ outgoing edges, the upper part of QRMDD($k$) is maximum when it is equivalent to a complete $2^k$-valued tree. Therefore, the upper part has at most

$$\frac{2^{sk}-1}{2^k-1}$$

nodes, where $s$ denotes the number of super variables in upper part. The lower part is maximum when all $i$-variable functions are realized for each level $i$ ($i = 1, 2, \ldots, u - s$), which have $2^k$-valued inputs and binary outputs. Note that $X_u$ may include dummy variables. Therefore, the number of nodes in a QRMDD($k$) is at most

$$\frac{2^{sk}-1}{2^k-1} + \sum_{i=1}^{u-s} 2^{2^{(ki-t)}}.$$

(Q.E.D.)

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