Multiple-Valued Decomposition of Generalized Boolean Functions and the Complexity of Programmable Logic Arrays

TSUTOMU SASAO, MEMBER, IEEE

***Abstract*—Generalized Boolean functions are shown to be useful for the design of programmable logic arrays (PLA's), and the complexity of three types of PLA's is obtained by the theory of multiple-valued decomposition. A two-level PLA consists of an AND array and an OR array, and they are cascaded to perform a two-level AND–OR circuit. A PLA with decoders consists of decoders, an AND array, and an OR array. A three-level PLA consists of a D array, an AND array, and an OR array, and they are cascaded to perform a three-level OR–AND–OR circuit. It is shown that a generalized Boolean function $f(X_1, X_2, \cdots, X_r)$: $\bigtimes_{i=1}^{r} B^{n_i} \rightarrow B$, where $B = \{0, 1\}$, is represented by a generalized Boolean expression of 2^{n_i}-valued variables X_i; and f can be directly realized by a PLA with decoders or a three-level PLA. To realize a function of n-variables ($n = 2r$), the following sizes are shown to be sufficient: for a two-level PLA, $(n + \frac{1}{2})2^n$; for a PLA with two-bit decoders, $\frac{1}{2}(n + \frac{1}{2})2^n$; for a three-level PLA, $2^n + (3n + 1)\sqrt{2}^n + 2n^2$. Especially in the case of PLA with two-bit decoders, the following sizes are shown to be necessary and sufficient: for an arbitrary symmetric function, $\frac{3}{2}(n + \frac{1}{2})\sqrt{3}^n$; and for a parity function, $(n + \frac{1}{2})\sqrt{2}^n$.**

***Index Terms*—Complexity of logic circuits, functional decomposition, multiple-valued logic, programmable logic array, symmetric function.**

I. INTRODUCTION

PROGRAMMABLE logic arrays (PLA's) are known as an approach to implement logic circuits having low production potential [1]–[4]. In this paper the complexity of three types of PLA's is discussed: a two-level PLA, a PLA with decoders, and a three-level PLA. The first type of PLA, a two-level PLA, is shown in Fig. 1. It consists of an AND array and an OR array. For example, the two-level PLA in Fig. 2 realizes the function of Table I. The AND array generates products of input variables, and the OR array generates sums of products. This PLA corresponds to a two-level AND–OR circuit. We define the size of this PLA as $C(n) = (2n + m)W$. The second type of PLA, a PLA with decoders, is shown in Fig. 3. Each decoder generates all the maxterms of its input variables. For example, the PLA in Fig. 4 realizes the function of Table I. We define the size of a PLA with decoders as $(H + m)W$. The two-level PLA can be considered as a special case of a PLA of this type, i.e., a PLA with one-bit decoders. The third type of PLA, a three-level PLA, is shown in Fig. 5. It is a PLA with a programmable input decoders. The D array generates the sum of input variables. This PLA corresponds to a three-level OR–AND–OR circuit. For example, the PLA in Fig. 6 realizes the function of Table I. We define the size of this PLA as $C(n) = (2n + W)H + Wm$.

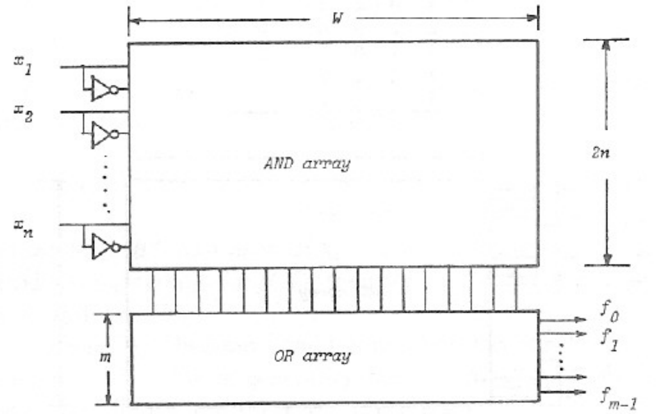

Fig. 1. Two-level PLA.

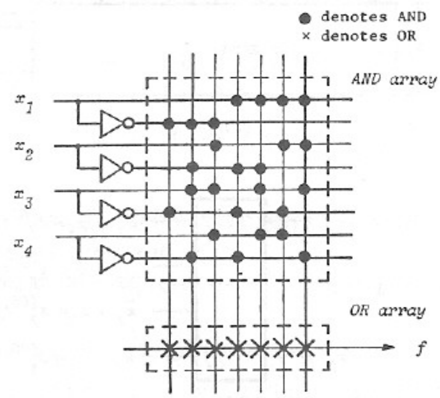

Fig. 2. Two-level PLA for Table I.

In Section II we introduce generalized Boolean functions and their expressions. They are useful for the design of PLA's with decoders or three-level PLA's. Generalized Boolean functions can be directly realized by PLA's with decoders or three-level PLA's.

In Section III we introduce the theory of multiple-valued decomposition of generalized Boolean functions. We use this theory to obtain the complexity of PLA's in Sections IV and V. Section III is somewhat involved and the reader may skip it for the first reading.

In Section IV, first we show that AND arrays of PLA's with decoders can be minimized by using generalized Boolean functions, then we obtain the size of PLA's with decoders for various classes of functions.

Manuscript received April 25, 1980; revised January 14, 1981 and February 18, 1981. This work was supported in part by the Ministry of Education of Japan under Grants 375187 and 475275.

The author is with the Department of Electrical Engineering, Osaka University, Osaka, Japan.

TABLE I
FOUR-VARIABLE FUNCTION

x_1	x_2	x_3	x_4	f
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

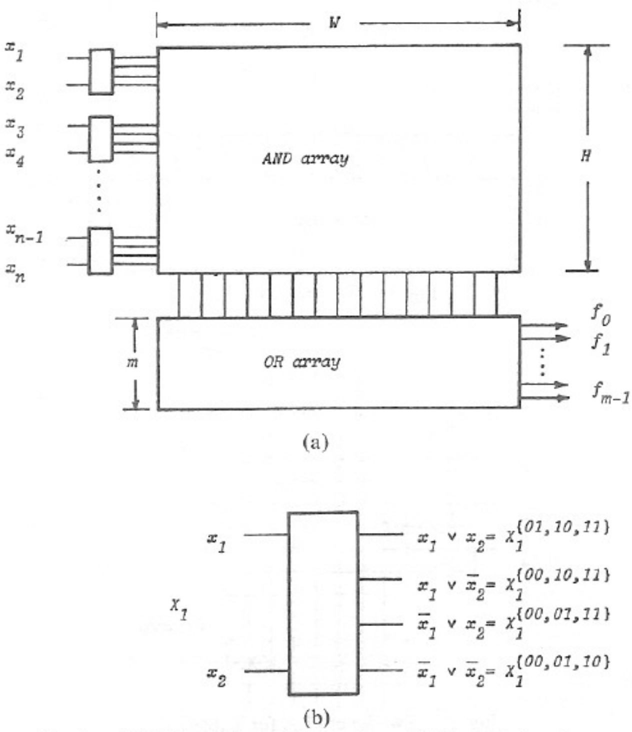

Fig. 3. (a) PLA with two-bit decoders. (b) Two-bit decoder.

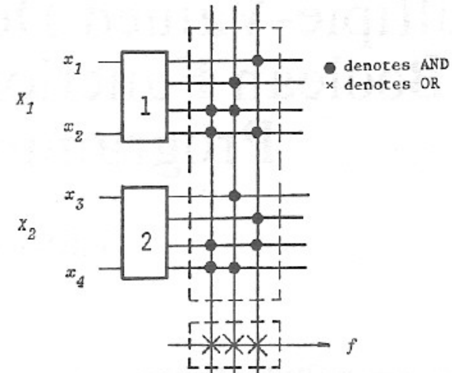

Fig. 4. PLA with two-bit decoders for Table I.

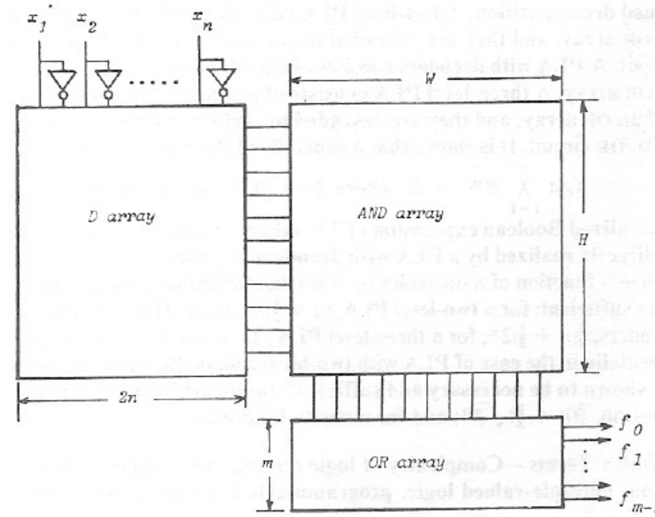

Fig. 5. Three-level PLA.

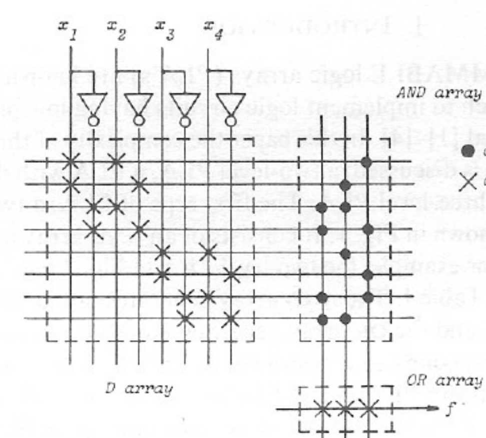

Fig. 6. Three-level PLA for Table I.

In Section V, first we show that AND arrays of three-level PLA's can be minimized by using generalized Boolean functions, then we obtain the size of three-level PLA's for various classes of functions.

Table II shows the sizes of PLA's for various classes of functions, which are obtained by using the theory of Section III. Table III shows the average size of PLA's for randomly generated functions, which are obtained by a computer experiment.

II. GENERALIZED BOOLEAN FUNCTION

In this section we introduce generalized Boolean functions and their expressions.

An ordinary function $f(x_1, x_2, \cdots, x_n)$: $\overbrace{B \times B \times \cdots \times B}^{n} \rightarrow B$, $B = \{0, 1\}$, can be represented by a Boolean expression of two-valued variables x_i $(i = 1, 2, \cdots, n)$, whereas the generalized Boolean function $f(X_1, X_2, \cdots, X_r)$: $B^{n_1} \times B^{n_2} \times \cdots \times B^{n_r} \rightarrow B$, can be represented by a generalized Boolean expression of 2^{n_i}-valued variables X_i $(i = 1, 2, \cdots, r)$.

Definition 1: Let $X = (x_1, x_2, \cdots, x_n)$ be a variable in B^n, where $B = \{0, 1\}$. The set of the variables in X is denoted by $\{X\}$. $(X_1, X_2, \cdots, X_r)$ is said to be a *partition* of X iff $\{X_1\} \cup \{X_2\} \cup \cdots \cup \{X_r\} = \{X\}$, $\{X_i\} \cap \{X_j\} = \phi$ $(i \neq j)$, and $\{X_i\} \neq \phi$ for all i and j. The number of the variables in $\{X_i\}$ is denoted by $d(X_i)$.

TABLE II
SIZES OF PLA'S WHICH ARE SUFFICIENT TO REALIZE n-VARIABLE FUNCTIONS

	Two-level PLA	PLA with two-bit decoders			Three-level PLA		
n	Parity function	Arbitrary function	Symmetric function	Parity function	Arbitrary function	Symmetric function	Parity function
6	416	208	117	52	288	258	196
8	2,176	1,088	459	136	784	572	392
10	10,752	5,376	1,701	336	2,216	1,168	680
12	51,200	25,600	6,075	800	6,752	2,377	1,032
14	237,568	118,784	21,141	1,856	22,280	4,670	1,600
16	1,081,344	540,672	72,171	4,224	78,592	8,740	2,320
n	$(n+\frac{1}{2})2^n$	$\frac{1}{2}(n+\frac{1}{2})2^n$	$\frac{2}{3}(n+\frac{1}{2})\sqrt{3}^n$	$(n+\frac{1}{2})\sqrt{2}^n$	(1)		

TABLE III
AVERAGE SIZE OF PLA'S FOR RANDOMLY GENERATED FUNCTIONS

	Two-level PLA					PLA with two-bit decoders				
	d: density					d:density				
n	10%	20%	30%	40%	50%	10%	20%	30%	40%	50%
6	75.4	109.2	148.2	170.3	185.9	63.7	88.4	117.0	128.7	132.6
8	321.3	537.2	661.3	759.9	770.1	280.5	464.1	535.5	578.0	564.4
10	1545.6	2478.0	3009.3	3234.0	3374.0	1383.0	2068.5	2349.9	2446.5	2554.0

"n" denotes the number of the external input variables. "d" denotes the percentage of input combinations which are mapped to 1.

* The entries of 40% and 50% of 10-variable function denote the average of 5 near minimal solutions; the other entries are the average of 10 minimal solutions.

Definition 2: Let $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be a constant in B^n. A symbol $X^{\mathbf{a}}$ denotes a mapping $X^{\mathbf{a}}: B^n \to B$, such that $X^{\mathbf{a}} = 0$ if $X \neq \mathbf{a}$ and $X^{\mathbf{a}} = 1$ if $X = \mathbf{a}$. Let $S \subseteq B^n$. A symbol X^S denotes the function such that

$$X^S = \bigvee_{\mathbf{a}_i \in S} X^{\mathbf{a}_i}.$$

Example 1: Let $X = (x_1, x_2)$, $\mathbf{a} = (0, 1)$ and $\mathbf{b} = (1, 0)$.

$$X^{\mathbf{a}} = \begin{cases} 1 & \text{if } X = (0, 1) \\ 0 & \text{if } X \neq (0, 1) \end{cases}, \text{ and}$$

$$X^{\{\mathbf{a},\mathbf{b}\}} = \begin{cases} 1 & \text{if } X = (0, 1) \quad \text{or } X = (1, 0) \\ 0 & \text{if } X = (0, 0) \quad \text{or } X = (1, 1). \end{cases}$$

For simplicity, $X^{\{(0,1),(1,0)\}}$ is denoted by $X^{\{01,10\}}$.

Lemma 1: Let $d(X) = n$ and $S_1, S_2 \subseteq B^n = I$.

$$X^{S_1} \cdot X^{S_2} = X^{S_1 \cap S_2},$$
$$X^{S_1} \vee X^{S_2} = X^{S_1 \cup S_2}, \overline{X^{S_1}} = X^{I - S_1}, X^I = 1, \text{ and } X^{\phi} = 0.$$

Let $\mathcal{B} = \{X^S : S \subseteq B^n\}$. The system $(\mathcal{B}, 0, 1, \vee, \cdot, -)$ is a 2^n-element Boolean algebra.

Example 2: Let $X = (x_1, x_2)$, $S_1 = \{00, 10\}$, and $S_2 = \{10, 11\}$.

$$X^{\{00,10\}} \cdot X^{\{10,11\}} = X^{\{10\}}, X^{\{00,10\}} \vee X^{\{10,11\}} = X^{\{00,10,11\}},$$
$$\overline{X^{\{00,10\}}} = X^{\{11,01\}}, X^{\{00,01,10,11\}} = 1, \text{ and } X^{\phi} = 0.$$

Definition 3: X^S is said to be a *literal*. A product of distinct literals is said to be a *term*. A sum of terms is said to be a *sum-of-products expression*. The number of terms in a sum-of-products expression P is denoted by $t(P)$. P is said to be *minimal* if there is no expression Q such that $t(Q) < t(P)$ and that Q denotes the same function as P.

Theorem 1: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X. An arbitrary function $f(X)$ can be represented in the form

$$\bigvee_{(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r)} f(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r) \cdot X_1^{\mathbf{a}_1} \cdot X_2^{\mathbf{a}_2} \cdot \cdots \cdot X_r^{\mathbf{a}_r} \qquad (1)$$

where $\mathbf{a}_i \in B^{n_i}$ and $n_i = d(X_i)$.

Example 3: 1) Let $X_1 = (x_1)$, $X_2 = (x_2)$, $X_3 = (x_3)$, and $X_4 = (x_4)$ be a (trivial) partition of $X = (x_1, x_2, x_3, x_4)$. The function of Table I can be represented as follows:

$$f(X_1, X_2, X_3, X_4) = X_1^0X_2^0X_3^0X_4^0 \vee X_1^0X_2^0X_3^0X_4^1 \vee X_1^0X_2^0X_3^1X_4^0$$
$$\vee X_1^0X_2^1X_3^0X_4^0 \vee X_1^0X_2^1X_3^0X_4^1 \vee X_1^0X_2^1X_3^1X_4^1 \vee X_1^1X_2^0X_3^0X_4^0$$
$$\vee X_1^1X_2^0X_3^1X_4^1 \vee X_1^1X_2^1X_3^0X_4^1 \vee X_1^1X_2^1X_3^1X_4^0.$$

2) Let $X = (X_1, X_2)$, $X_1 = (x_1, x_2)$, and $X_2 = (x_3, x_4)$ be a partition of $X = (x_1, x_2, x_3, x_4)$. The function of Table I can be represented as follows:

$$f(X_1, X_2) = X_1^{(00)} \cdot X_2^{(00)} \vee X_1^{(00)} \cdot X_2^{(01)} \vee X_1^{(00)} \cdot X_2^{(10)}$$
$$\vee X_1^{(01)} \cdot X_2^{(00)} \vee X_1^{(01)} \cdot X_2^{(01)} \vee X_1^{(01)} \cdot X_2^{(11)}$$
$$\vee X_1^{(10)} \cdot X_2^{(00)} \vee X_1^{(10)} \cdot X_2^{(11)} \vee X_1^{(11)} \cdot X_2^{(01)} \vee X_1^{(11)} \cdot X_2^{(10)}.$$

Theorem 2: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X. An arbitrary function can be represented in a form

$$f(X_1, X_2, \cdots, X_r) = \bigvee_{(S_1, S_2, \cdots, S_r)} X_1^{S_1} \cdot X_2^{S_2} \cdots X_r^{S_r} \qquad (2)$$

where $S_i \subseteq B^{n_i}$, and $n_i = d(X_i)$. If P is a minimal sum-of-products expression for $f(X)$, then $t(P) \leq 2^{n - \max_i\{n_i\}}$, where $n_i = d(X_i)$.

Proof: By Theorem 1 and Lemma 1 we have the first part. Assume without loss of generality that $n_1 = \max_i \{n_i\}$. $f(X)$ can be represented as

$$f(X_1, X_2, \cdots, X_r)$$
$$= \bigvee_{(\mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_r)} X_1^{S_1} \cdot X_2^{\mathbf{a}_2} \cdot X_3^{\mathbf{a}_3} \cdots X_r^{\mathbf{a}_r}. \qquad (3)$$

The number of terms in (3) is at most $\prod_{i=2}^{n} 2^{n_i} = 2^{n - n_1} = 2^{n - \max_i\{n_i\}}$. Q.E.D.

The expression which has the form (2) is called *a generalized Boolean expression.*

Example 4: The expressions of Example 3 can be simplified as follows:

1) $$f(X_1, X_2, X_3, X_4) = X_1^0X_2^{\{0,1\}} \cdot X_3^0X_4^{\{0,1\}}$$
$$\vee X_1^0X_2^0X_3^1X_4^0 \vee X_1^0X_2^1X_3^1X_4^1 \vee X_1^1X_2^0X_3^0X_4^0$$
$$\vee X_1^1X_2^0X_3^1X_4^1 \vee X_1^1X_2^1X_3^0X_4^1 \vee X_1^1X_2^1X_3^1X_4^0 \qquad (4)$$

2) $$f(X_1, X_2) = X_1^{\{00,01\}} \cdot X_2^{\{00,01\}}$$
$$\vee X_1^{\{00,11\}} \cdot X_2^{\{01,10\}} \vee X_1^{\{01,10\}} \cdot X_2^{\{00,11\}}. \qquad (5)$$

1) is a trivial case of generalized Boolean expression and essentially the same as the ordinary one. However, 2) is a nontrivial case of generalized Boolean function. Note that the number of terms in (5) is smaller than that of (4).

III. MULTIPLE-VALUED DECOMPOSITION OF GENERALIZED BOOLEAN FUNCTIONS

In Section III-A we introduce multiple-valued decomposition of generalized Boolean functions. In Section III-B we obtain the number of terms in a generalized Boolean expression for various classes of functions.

A. Decomposition Theory

Definition 4: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X, and $f(X)$ be a function such that

$$f: B^{n_1} \times B^{n_2} \times \cdots \times B^{n_r} \to B.$$

For $\boldsymbol{a}, \boldsymbol{b} \in B^{n_i}$, define a relation $\underset{\sim}{i}$ such that

$$\boldsymbol{a} \underset{\sim}{i} \boldsymbol{b} \Leftrightarrow f(X|X_i = \boldsymbol{a}) = f(X|X_i = \boldsymbol{b})$$

where $f(X|X_i = \boldsymbol{a})$ denotes $f(X_1, X_2, \cdots, X_{i-1}, \boldsymbol{a}, X_{i+1}, \cdots, X_r)$. Obviously, the relation $\underset{\sim}{i}$ is an equivalence relation. Let $\Pi_i = (L_0^i, L_1^i, \cdots, L_{k_i-1}^i)$ be a partition of B^{n_i} induced by the equivalence relation $\underset{\sim}{i}$. A function $\Psi_i : B^{n_i} \to M_i$; $M_i = \{0, 1, \cdots, k_i - 1\}$ such that $\Psi_i(\boldsymbol{a}) = j \Leftrightarrow \boldsymbol{a} \in L_j^i$ is called a partition function of B^{n_i}, where $1 \le k_i \le 2^{n_i}$.

Example 5: Consider a six variable function

$$f(X) = (\bar{x}_1 \vee \bar{x}_2) \cdot (x_3 \oplus x_4) \cdot (\bar{x}_5 \vee \bar{x}_6) \vee (x_1 \vee x_2) \cdot (x_3 \oplus \bar{x}_4) \cdot x_5 \vee (x_1 \oplus x_2) \cdot (x_5 \oplus x_6).$$

Let (X_1, X_2, X_3) be a partition of X, where $X_1 = (x_1, x_2)$, $X_2 = (x_3, x_4)$, and $X_3 = (x_5, x_6)$. The function $f(X)$ has the properties that

$$f(X|X_1 = (01)) = f(X|X_1 = (10)), f(X|X_2 = (00)) = f(X|X_2 = (11)), \text{and } f(X|X_2 = (01)) = f(X|X_2 = (10)).$$

Partitions on B^{n_i} induced by the equivalence relations $\underset{\sim}{i}$ ($i = 1, 2, 3$) are $\Pi_1 = ([00], [01, 10], [11])$, $\Pi_2 = ([00, 11], [01, 10])$, and $\Pi_3 = ([00], [01], [10], [11])$, respectively. The partition functions of B^{n_i} are shown in Table IV.

Lemma 2: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X, $d(X_i) = n_i$, and let Ψ_i be a partition function of B^{n_i}. There exists a multiple-valued input two-valued output function g: $M_1 \times M_2 \times \cdots \times M_r \to B$ such that $f(X_1, X_2, \cdots, X_r) = g(\Psi_1(X_1), \Psi_2(X_2), \cdots, \Psi_r(X_r))$, where $1 \le |M_i| = k_i \le 2^{n_i}$.

Proof: For each $b_i \in M_i$, there exists $\boldsymbol{a}_i$ such that $\Psi_i(\boldsymbol{a}_i) = b_i$ ($i = 1, 2, \cdots, r$). Let $g(b_1, b_2, \cdots, b_r) = f(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_r)$. It is easy to show that this function satisfies the condition of the theorem. Q.E.D.

This lemma is similar to the well-known decomposition theorem of Ashenhurst [5]. But, Ψ_i is, in general, a multiple-valued function. When $M_i = \{0, 1\}$, ($i = 1, 2, \cdots, r$), this lemma reduces to the ordinary decomposition theorem.

Example 6: Consider the function of Example 5. By Lemma 2 $f(X)$ can be represented as $f(X_1, X_2, X_3) = g(\Psi_1(X_1), \Psi_2(X_2), \Psi_3(X_3))$, where $g(Y_1, Y_2, Y_3)$ is shown in Table V.

Definition 5: Let $M = \{0, 1, \cdots, k - 1\}$, $t \in M$, and Y^t: $M \to B$ be a function such that $Y^t = 0$ if $Y \ne t$ and $Y^t = 1$ if $Y = t$. Let $T \subseteq M$, Y^T denotes a function such that $Y^T = \bigvee_{t \in Y} Y^t$.

Lemma 3: Let $T_1, T_2 \subseteq M = I$.

$$Y^{T_1} \cdot Y^{T_2} = Y^{T_1 \cap T_2}, Y^{T_1} \vee Y^{T_2} = Y^{T_1 \cup T_2}, \overline{Y^{T_1}} = Y^{I - T_1}, Y^I = 1, \text{and } Y^\phi = 0.$$

Let $\mathcal{B} = \{Y^T : T \subseteq M\}$. The system $(\mathcal{B}, 0, 1, \vee, \cdot, -)$ is a Boolean algebra.

Lemma 4: A multiple-valued input two-valued output function g: $M_1 \times M_2 \times \cdots \times M_r \to B$ can be represented in the form

$$g(Y_1, Y_2, \cdots, Y_r) = \bigvee_{(t_1, t_2, \cdots, t_r)} g(t_1, t_2, \cdots, t_r)\, Y_1^{t_1} \cdot Y_2^{t_2} \cdots Y_r^{t_r} \quad (6)$$

TABLE IV
PARTITION FUNCTION

X_i	$\psi_1(X_1)$	$\psi_2(X_2)$	$\psi_3(X_3)$
00	0	0	0
01	1	1	1
10	1	1	2
11	2	0	3

TABLE V
EXAMPLE 5

Y_1	Y_2	Y_3	g
0	0	0	0
0	0	1	0
0	0	2	0
0	0	3	0
0	1	0	1
0	1	1	1
0	1	2	1
0	1	3	0
1	0	0	0
1	0	1	1
1	0	2	1
1	0	3	1
1	1	0	1
1	1	1	1
1	1	2	1
1	1	3	0
2	0	0	0
2	0	1	0
2	0	2	1
2	0	3	1
2	1	0	0
2	1	1	0
2	1	2	0
2	1	3	0

or in a form

$$g(Y_1, Y_2, \cdots, Y_r) = \bigvee_{(T_1, T_2, \cdots, T_r)} Y_1^{T_1} \cdot Y_2^{T_2} \cdots Y_r^{T_r} \quad (7)$$

where $t_i \in M_i$, $T_i \subseteq M_i$, and $M_i = \{0, 1, \cdots, k_i - 1\}$.

Proof: By Definition 5 it is easy to show that (6) holds. By Lemma 3 and (6) we have (7). Q.E.D.

Example 7: The function g of Table V can be represented in the form

$$g(Y_1, Y_2, Y_3) = Y_1^0 Y_2^1 Y_3^0 \vee Y_1^0 Y_2^1 Y_3^1 \vee Y_1^0 Y_2^1 Y_3^2 \vee Y_1^1 Y_2^0 Y_3^1 \vee Y_1^1 Y_2^0 Y_3^2 \vee Y_1^1 Y_2^0 Y_3^3 \vee Y_1^1 Y_2^1 Y_3^0 \vee Y_1^1 Y_2^1 Y_3^1 \vee Y_1^1 Y_2^1 Y_3^2 \vee Y_1^2 Y_2^0 Y_3^2 \vee Y_1^2 Y_2^0 Y_3^3$$

or in a form

$$g(Y_1, Y_2, Y_3) = Y_1^{\{0,1\}} \cdot Y_2^1 \cdot Y_3^{\{0,1,2\}} \vee Y_1^{\{1,2\}} \cdot Y_2^0 \cdot Y_3^{\{2,3\}} \vee Y_1^1 \cdot Y_3^{\{1,2\}}. \quad (8)$$

B. Number of Terms

Lemma 5: Let f, Ψ_i, and g be the functions of Lemma 2. A literal $Y_i^{T_i}$ of the expression (7) corresponds to the literal $X_i^{S_i}$ of (2), where $S_i = \Psi_i^{-1}(T_i)$. And a term $Y_1^{T_1} \cdot Y_2^{T_2} \cdots Y_r^{T_r}$ of (7) corresponds to the term $X_1^{S_1} \cdot X_2^{S_2} \cdots X_r^{S_r}$ of (2).

Proof: It is easy to show by Definitions 4 and 5. Q.E.D.

Example 8: Consider the function $f(X)$ of Example 5 and the function $g(Y)$ of Example 7. For the term $Y_1^{\{0,1\}} \cdot Y_2^1 \cdot Y_3^{\{0,1,2\}}$ of $g(Y)$, the corresponding term of $f(X)$ is $X_1^{\{00,01,10\}} \cdot X_2^{\{01,10\}} \cdot X_3^{\{00,01,10\}}$. For the term $Y_1^{\{1,2\}} \cdot Y_2^0 \cdot Y_3^{\{2,3\}}$, the corresponding term of $f(X)$ is

$$X_1^{\{01,10,11\}} \cdot X_2^{\{00,11\}} \cdot X_3^{\{10,11\}}.$$

And for the term $Y_1^1 Y_3^{\{1,2\}}$, the corresponding term of $f(X)$ is $X_1^{\{01,10\}} \cdot X_3^{\{01,10\}}$.

Theorem 3: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X. Suppose that f and g satisfy the relation

$$f(X_1, X_2, \cdots, X_r) = g(\Psi_1(X_1), \Psi_2(X_2), \cdots, \Psi_r(X_r)).$$

If P_1 and Q_1 are minimal sum-of-products expressions for f and g, respectively, then $t(P_1) = t(Q_1) \le \left(\prod_{i=1}^{r} k_i\right) / \left(\max_i \{k_i\}\right)$, where $\Psi_i: B_i^{n_i} \to M_i$; $M_i = \{0, 1, \cdots, k_i - 1\}$, $n_i = d(X_i)$, and $1 \le |M_i| = k_i \le 2^{n_i}$.

Proof: 1) For P_1, a minimal sum-of-products expression of $f(X)$

$$f(X_1, X_2, \cdots, X_r) = \bigvee_{(S_1, S_2, \cdots, S_r)} X_1^{S_1} \cdot X_2^{S_2} \cdots X_r^{S_r}$$

consider the expression P_2, which has the form

$$\bigvee_{(G_1, G_2, \cdots, G_r)} Y_1^{G_1} \cdot Y_2^{G_2} \cdots Y_r^{G_r}, \qquad \text{where } G_i = \Psi_i(S_i).$$

Clearly, $t(P_1) = t(P_2)$. It is easy to show that P_2 represents $g(Y)$. For Q_1, a minimal sum-of-products expression of $g(Y)$

$$g(Y_1, Y_2, \cdots, Y_r) = \bigvee_{(T_1, T_2, \cdots, T_r)} Y_1^{T_1} \cdot Y_2^{T_2} \cdots Y_r^{T_r}$$

consider the expression Q_2, which has the form

$$\bigvee_{(D_1, D_2, \cdots, D_r)} X_1^{D_1} \cdot X_2^{D_2} \cdots X_r^{D_r}, \qquad \text{where } D_i = \Psi_i^{-1}(T_i).$$

Clearly, $t(Q_1) = t(Q_2)$. By Lemma 5, Q_2 represents $f(X)$. As P_1 is a minimal expression of $f(X)$, we have $t(P_1) \le t(Q_2)$. As Q_1 is a minimal expression of $g(Y)$, we have $t(Q_1) \le t(P_2)$. Therefore, $t(P_1) = t(Q_1)$.

2) Assume without loss of generality that $\max\{k_i\} = k_1$. $g(Y)$ can be represented in the form

$$g(Y_1, Y_2, \cdots, Y_r) = \bigvee_{(t_2, \cdots, t_r)} Y_1^{T_1} \cdot Y_2^{t_2} \cdot Y_3^{t_3} \cdots Y_r^{t_r} \qquad (9)$$

where $T_1 \subseteq M_1$ and $t_i \in M_i$ $(i = 2, 3, \cdots, r)$. The number of terms in (9) is at most

$$\prod_{i=2}^{r} k_i = \left(\prod_{i=1}^{r} k_i\right) \Big/ \left(\max_i \{k_i\}\right). \qquad \text{Q.E.D.}$$

Example 9: The expression (8) of Example 4 is a minimal sum-of-products expression for $g(Y)$. Therefore, the corresponding minimal expression for $f(X)$ is given by

$$f(X_1, X_2, X_3) = X_1^{\{00,01,10\}} \cdot X_2^{\{01,10\}} \cdot X_3^{\{00,01,10\}} \vee X_1^{\{01,10,11\}} \cdot X_2^{\{00,11\}} \cdot X_3^{\{10,11\}} \vee X_1^{\{01,10\}} \cdot X_3^{\{01,10\}}.$$

Theorem 4: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X. Suppose that f can be represented in a form $f(X_1, X_2, \cdots, X_r) = g(\Psi_1(X_1), \Psi_2(X_2), \cdots, \Psi_r(X_r))$, where $\Psi_i: B^{n_i} \to M_i$; $M_i = \{0, 1, \cdots, k_i - 1\}$; $1 \le k_i = |M_i| \le 2^{n_i}$ is a partition function. There exists a function $f(X_1, X_2, \cdots, X_r)$ whose minimal sum-of-products expression contains $\left(\prod_{i=1}^{r} k_i\right) / \left(\max_i \{k_i\}\right)$ terms.

Proof: Assume without loss of generality that $\max\{k_i\} = k_1$. Let $g(Y)$ be

$$g(Y_1, Y_2, \cdots, Y_r) = \begin{cases} 1 & \text{if } Y_1 + Y_2 + \cdots + Y_r = 0 \pmod{k_1} \\ 0 & \text{otherwise} \end{cases}$$

$g(Y)$ can be written as

$$g(Y_1, Y_2, \cdots, Y_r) = \bigvee_{\substack{t_1 + t_2 + \cdots t_r = 0 \\ (\text{mod} k_1)}} Y_1^{t_1} \cdot Y_2^{t_2} \cdots Y_r^{t_r}$$

$$t_i \in \{0, 1, \cdots, k_i - 1\}. \qquad (10)$$

Every expression for $g(Y)$ has the form (10) because if the expression has a term of the form

$$Y_1^{t_1} \cdot Y_2^{t_2} \cdots Y_i^{S_i} \cdots Y_r^{t_r}, \qquad |S_i| \ge 2$$

then it cannot satisfy the condition for $g(Y)$. Therefore, the minimal expression which represents $g(Y)$ has the form (10). For arbitrary $t_2, t_3, \cdots, t_r$, there exists $t_1 \in \{0, 1, \cdots, k_1 - 1\}$ such that $t_1 + t_2 + \cdots + t_r = 0 \pmod{k_1}$. So the number of the terms of (10) is $\prod_{i=2}^{r} k_i$. By Theorem 3 the number of the terms of f is also $\prod_{i=2}^{r} k_i$. Q.E.D.

Theorem 5: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X, $d(X_i) = q$, and $qr = n$. The necessary and sufficient number of terms to represent an arbitrary function $f(X_1, X_2, \cdots, X_r)$ is 2^{n-q}.

Proof—Sufficiency: By Theorem 2 the minimal sum-of-products expression for f contains at most 2^{n-q} terms.

Necessity: Let the function $f(X)$ be decomposed as

$$f(X_1, X_2, \cdots, X_r) = g(\Psi_1(X_1), \Psi_2(X_2), \cdots, \Psi_r(X_r))$$

where

$$\Psi_i(X_i) = \sum_{k=1}^{q} x_{i_k} 2^{q-k}, X_i = (x_{i_1}, x_{i_2}, \cdots, x_{i_q})$$

and let Ψ_i denote a partition function $\Psi_i: B^q \to M$; $M = \{0, 1, \cdots, 2^q - 1\}$. By Theorem 4 there exists a function whose minimal sum-of-products expression contains $(2^q)^{r-1}$ terms. Q.E.D.

Example 10: Consider the function f_1 in Table VI. Let $X_1 = (x_1, x_2)$ and $X_2 = (x_3, x_4)$. f_1 can be decomposed as $f_1(X_1, X_2) = g_1(\Psi_1(X_1), \Psi_2(X_2))$, where $\Psi_i(00) = 0$, $\Psi_i(01) = 1$, $\Psi_i(10) = 2$, $\Psi_i(11) = 3$, $(i = 1, 2)$, and

$$g_1(Y_1, Y_2) = \begin{cases} 1 & Y_1 + Y_2 = 0 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

The minimal sum-of-products expression for $g_1(Y_1, Y_2)$ is

$$g_1(Y_1, Y_2), = Y_1^0 \cdot Y_2^0 \vee Y_1^1 \cdot Y_2^3 \vee Y_1^2 \cdot Y_2^2 \vee Y_1^3 \cdot Y_2^1,$$

and the corresponding expression for $f(X_1, X_2)$ is

$$f_1(X_1, X_2) = X_1^{(00)} \cdot X_2^{(00)} \vee X_1^{(01)} \cdot X_2^{(11)} \vee X_1^{(10)} \cdot X_2^{(10)} \vee X_1^{(11)} \cdot X_2^{(01)}.$$

TABLE VI

x_1	x_2	x_3	x_4	f_1	f_2	f_3
0	0	0	0	1	1	1
0	0	0	1	0	0	0
0	0	1	0	0	0	0
0	0	1	1	0	0	1
0	1	0	0	0	0	0
0	1	0	1	0	0	1
0	1	1	0	0	0	1
0	1	1	1	1	1	0
1	0	0	0	0	0	0
1	0	0	1	0	0	1
1	0	1	0	1	0	1
1	0	1	1	0	1	0
1	1	0	0	0	0	1
1	1	0	1	1	1	0
1	1	1	0	0	1	0
1	1	1	1	0	0	1

Therefore, the minimal sum-of-products expression for f_1 contains $2^{n-q} = 4$ terms. This is an example of a function of Theorem 5, where $r = q = 2$.

Definition 6: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X. $f(X)$ is said to be *partially symmetric* with respect to X_i if $f(X)$ is invariant under any permutation of variables in $\{X_i\}$.

Theorem 6: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X, $d(X_i) = q$ and $qr = n$. Let $f(X)$ be partially symmetric with respect to X_i for all $i (i = 1, 2, \cdots, r)$. The necessary and sufficient number of terms to represent f is $(q + 1)^{r-1}$.

Proof—Sufficiency: By the property of the partially symmetric function, $f(X)$ can be decomposed as

$$f(X_1, X_2, \cdots, X_r) = g(\Psi_1(X_1), \Psi_2(X_2), \cdots, \Psi_r(X_r)) \quad (11)$$

where $\Psi_i(X_i) = |X_i|$ $(0 \le |X_i| \le q$, $|X_i|$ denotes the number of 1's in $X_i)$. By Theorem 3 the minimal sum-of-products expression for f contains at most $(q + 1)^{r-1}$ terms.

Necessity: By Theorem 4 there exists a function whose minimal sum-of products expression contains $(q + 1)^{r-1}$ terms. Q.E.D.

Example 11: Consider the function f_2 in Table VI. Let $X_1 = (x_1, x_2)$ and $X_2 = (x_3, x_4)$. f_2 is partially symmetric with respect to X_i, and it can be decomposed as

$$f_2(X_1, X_2) = g_2(\Psi_1(X_1), \Psi_2(X_2)),$$
where $\Psi_i(00) = 0$, $\Psi_i(01) = \Psi_i(10) = 1$, $\Psi_i(11) = 2$, $(i = 1, 2)$ and

$$g_2(Y_1, Y_2) = \begin{cases} 1 & \text{if } Y_1 + Y_2 = 0 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

The minimal sum-of-products expression for f_2 is

$$g_2(Y_1, Y_2) = Y_1^0 \cdot Y_2^0 \vee Y_1^1 \cdot Y_2^2 \vee Y_1^2 \cdot Y_2^1$$

and the corresponding expression for f_2 is

$$f_2(X_1, X_2) = X_1^{\{00\}} \cdot X_2^{\{00\}} \vee X_1^{\{01,10\}} \cdot X_2^{\{11\}} \vee X_1^{\{11\}} \cdot X_2^{\{01,10\}}.$$

Therefore, the minimal sum-of-products expression for f_2 contains $(q + 1)^{r-1} = 3$ terms. This is an example of a function of Theorem 6, where $q = r = 2$.

Theorem 7: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X, and $d(X_i) = q$. Let $f(X)$ be a parity function of n variables. The number of terms in the minimal sum-of-products expression for f is 2^{r-1}.

Proof: By the property of a parity function, $f(X)$ can be decomposed as $f(X_1, X_2, \cdots, X_r) = g(\Psi_1(X_1), \Psi_2(X_2), \cdots, \Psi_r(X_r))$, where

$$\Psi_i(X_i) = \begin{cases} 1 & \text{if } |X_i| = 1 \pmod 2 \\ 0 & \text{if } |X_i| = 0 \pmod 2. \end{cases}$$

By Theorem 3 the minimal sum-of-products expression of f contains at most 2^{r-1} terms. Clearly, $\Psi_i: B^{n_i} \to \{0, 1\}$ is a partition function. By Theorem 4 we need 2^{r-1} terms to represent f. Q.E.D.

Example 12: Consider the function f_3 in Table VI. Let $X_1 = (x_1, x_2)$ and $X_2 = (x_3, x_4)$. f_3 is parity function and it can be decomposed as $f_3(X_1, X_2) = g_3(\Psi_1(X_1), \Psi_2(X_2))$, where $\Psi_i(00) = \Psi_i(11) = 0$, $\Psi_i(01) = \Psi_i(10) = 1$, and

$$g_3(Y_1, Y_2) = \begin{cases} 1 & \text{if } Y_1 + Y_2 = 0 \pmod 2 \\ 0 & \text{if } Y_1 + Y_2 = 1 \pmod 2. \end{cases}$$

The minimal sum-of-products expression for $g_3(Y_1, Y_2)$ is

$$g_3(Y_1, Y_2) = Y_1^0 \cdot Y_2^0 \vee Y_1^1 \cdot Y_2^1$$

and the corresponding expression for $f(X_1, X_2)$ is

$$f_3(X_1, X_2) = X_1^{\{00,11\}} \cdot X_2^{\{00,11\}} \vee X_1^{\{01,10\}} \cdot X_2^{\{01,10\}}.$$

Therefore, the minimal sum-of-products expression for f_3 contains $2^{r-1} = 2$ terms. This is an example of a function of Theorem 7, where $q = r = 2$.

IV. Complexity of Programmable Logic Arrays with Decoders

In Section IV-A we show that a PLA with decoders realizes a generalized Boolean function. In Section IV-B we obtain the sizes of PLA's for various classes of functions.

A. Realization of Functions Using PLA's with Decoders

Definition 7: Let $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be a constant in B^n. $\overline{X^{\mathbf{a}}}$ is said to be a *maxterm* of X.

By Definition 2 we have the following.

Lemma 6: X^S can be represented by the product of some maxterms of X

$$X^S = \bigwedge_{\mathbf{a}_i \in (B^n - S)} \overline{X^{\mathbf{a}_i}}.$$

Theorem 8: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X. In a PLA with decoders, if each decoder generates all the maxterms of X_i for $i = 1, 2, \cdots, r$, then an arbitrary term which has the form $X_1^{S_1} \cdot X_2^{S_2} \cdots X_r^{S_r}$ can be realized in each column of the AND array. The width W of the AND array to realize the given function f is equal to the number of terms in a minimal sum-of-products expression for f and $W \le 2^{n-\max_i\{n_i\}}$.

Example 13—a) In the Case of the Two-Level PLA: Let $X_1 = (x_1)$, $X_2 = (x_2)$, $X_3 = (x_3)$, and $X_4 = (x_4)$ be a (trivial) partition of $X = (x_1, x_2, x_3, x_4)$. Fig. 2 shows the two-level PLA for the function of Table I. Each column of the AND array corresponds to each term of (4).

b) In the Case of the PLA with Two-Bit Decoders: Let $X = (X_1, X_2)$, $X_1 = (x_1, x_2)$, and $X_2 = (x_3, x_4)$ be a partition of $X = (x_1, x_2, x_3, x_4)$. Fig. 4 shows the PLA with two-bit decoders for the function of Table I. Each column of the AND array corresponds to each term of (5). (End of the example.)

By Theorem 8, to minimize the size of the AND array for $f(X)$, it is sufficient to obtain a minimal sum-of-products expression of $f(X)$ having the form (2). In the case of a two-level PLA, the expression denotes the function f: $\{0, 1\}^n \rightarrow \{0, 1\}$, whereas in the case of a PLA with two-bit decoders, the expression denotes the two-valued function of four-valued variables f: $\{00, 01, 10, 11\}^{n/2} \rightarrow \{0, 1\}$.

Theorem 9: Let W_1 and W_2 be the width of the two-level PLA and the PLA with two-bit decoders to realize a function, respectively. Then $W_1 \leq W_2$.

B. Formulas for PLA's with Decoders

Theorem 10: In a PLA with q-bit decoders the following size is necessary and sufficient to realize an n-variable function, when the assignment of the input variables is fixed to the decoders where $n = qr$.

1) *For an Arbitrary Function:* $(r \cdot 2^q + 1) \cdot 2^{n-q}$.
2) *For an Arbitrary Symmetric Function:* $(r \cdot 2^q + 1) \cdot (q + 1)^{r-1}$.
3) *For a Parity Function:* $(r \cdot 2^q + 1) \cdot 2^{r-1}$.

Proof: By Theorems 5–7. Q.E.D.

Corollary 1: In a PLA with two-bit decoders the following size is necessary and sufficient to realize an n-variable function, when the assignment of the input variables is fixed to the decoders ($n = 2r$).

1) *For an Arbitrary Function:* $\frac{1}{2}(n + \frac{1}{2}) \cdot 2^n$.
2) *For an Arbitrary Symmetric Function:* $\frac{2}{3}(n + \frac{1}{2})\sqrt{3}^n$.
3) *For a Parity Function:* $(n + \frac{1}{2})\sqrt{2}^n$.

Corollary 2: The size of two-level PLA to realize a parity function of n variable is $(n + \frac{1}{2})2^n$.

C. Assignment Problem of the Input Variables to the Decoders

In the case of the PLA with decoders the way of assignment of the input variables to the decoders often influences the size of the PLA.

Example 14: Let us realize the function of Table I by using a PLA with two-bit decoders. Assume that $X = (X_1, X_2)$ is a partition of the input variables X. There exist three possible ways of assignment of four input variables to two two-bit decoders.

1) When the input variables are assigned as $X_1 = (x_1, x_2)$ and $X_2 = (x_3, x_4)$. (See Fig. 4.) The minimal sum-of-products expression is

$$f(X_1, X_2) = X_1^{\{00,01\}} \cdot X_2^{\{00,01\}} \vee X_1^{\{00,11\}} \cdot X_2^{\{01,10\}} \vee X_1^{\{01,10\}} \cdot X_2^{\{00,11\}}.$$

So three columns are necessary in this assignment.

2) When the input variables are assigned as $X_1 = (x_1, x_3)$ and $X_2 = (x_2, x_4)$. (See Fig. 7.) The minimal sum-of-products expression is

$$f(X_1, X_2) = X_1^{\{00,01,10\}} \cdot X_2^{\{00,11\}} \vee X_1^{\{00,11\}} \cdot X_2^{\{01,10\}}.$$

So two columns are necessary in this assignment.

3) When the input variables are assigned as $X_1 = (x_1, x_4)$ and $X_2 = (x_2, x_3)$. (See Fig. 8.) The minimal sum-of-products expression is

$$f(X_1, X_2) = X_1^{\{00,11\}} \cdot X_2^{\{01,10\}} \vee X_1^{\{01,10\}} \cdot X_2^{\{00,11\}} \vee X_1^{\{00,01\}} \cdot X_2^{\{00,10\}}.$$

So three columns are necessary in this assignment.

Therefore, when the input variables are assigned as shown in Fig. 7, the array is the minimum.

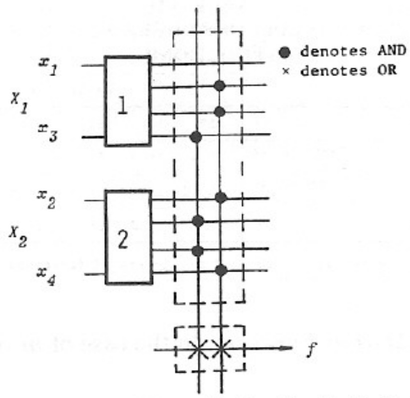

Fig. 7. PLA with two-bit decoders for Table I.

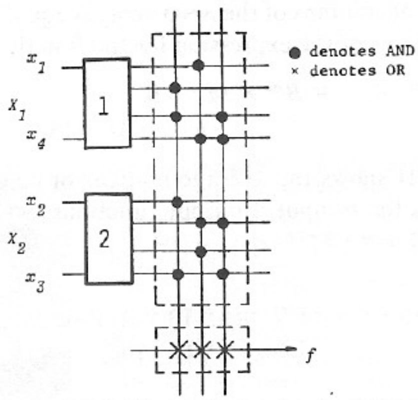

Fig. 8. PLA with two-bit decoders for Table I.

Example 15: Consider the function f_1 in Table VI. When the input variables are assigned as $X_1 = (x_1, x_3)$ and $X_2 = (x_2, x_4)$, $f_1(X_1, X_2)$ can be written as

$$\begin{aligned} f_1(X_1, X_2) &= X_1^{(00)} \cdot X_2^{(00)} \vee X_1^{(01)} \cdot X_2^{(11)} \\ &\quad \vee X_1^{(11)} \cdot X_2^{(00)} \vee X_1^{(10)} \cdot X_2^{(11)} \\ &= X_1^{\{00,11\}} \cdot X_2^{(00)} \vee X_1^{\{01,10\}} \cdot X_2^{(11)}. \end{aligned}$$

Therefore, the function f_1 requires only two terms in this assignment.

D. Statistical Results

Average Size of PLA's with Two-Bit Decoders: Table III shows the average size of PLA's for up to 10-variable functions. $d = (u/2^n) \times 100$ denotes the percentage of minterms which are mapped to one.

The Effect of the Assignment of Input Variables to the Decoders: To investigate the dependence on the way of assignment of input variables, ten functions of 8 variables were randomly generated for each density. Then 105 expressions which correspond to all possible ways of assignments were minimized. Table VII shows the statistical result of this exhaustive investigation.

TABLE VII
AVERAGE NUMBER OF COLUMNS OF PLA'S FOR EIGHT-VARIABLE FUNCTIONS

Type of PLA		density: d (%) 5%	10%	15%	20%	30%	40%
PLA's with two-bit decoders	Assignment is optimal	8.4	14.7	19.9	24.2	24.8	30.2
	Assignment is non-optimal*	10.10	17.27	22.71	27.80	31.77	33.58
Two-level PLA's		10.8	19.2	26.5	33.1	39.5	44.4

Each entry is the average of 10 randomly generated functions.
* The average of 105 assignments.

Multiple-Output Function: In the case of m-output function

$$f_j : B^{n_1} \times B^{n_2} \times \cdots \times B^{n_r} \rightarrow B \qquad (j = 0, 1, \cdots, m-1)$$

the number of columns of the AND array is equal to the number of the terms of the expression for the function

$$\mathcal{F}: B^{n_1} \times B^{n_2} \times \cdots \times B^{n_r} \times M \rightarrow B,$$
$$\text{where } M = \{0, 1, \cdots, m-1\}.$$

Table VIII shows the average number of columns of the AND arrays for n-input 4-output functions, where $d = u/(2^n \cdot m)$ and $u = |\mathcal{F}^{-1}(1)|$.

TABLE VIII
AVERAGE NUMBER OF COLUMNS OF PLA'S FOR n-INPUT 4-OUTPUT FUNCTIONS

n	Two-level PLA density 12.5%	25.0%	37.5%	50.0%	PLA with two-bit decoders density 12.5%	25.0%	37.5%	50.0%
4	6.2	9.2	10.8	12.0	5.8	7.8	9.4	10.6
6	21.6	35.2	42.0	41.8	19.4	31.2	34.2	33.6
8	86.4	123.0	140.0	150.6	76.4	107.6	118.6	123.0

* Average of 5 randomly generated functions.

V. COMPLEXITY OF THREE-LEVEL PROGRAMMABLE LOGIC ARRAYS

In this section, first we show that a three-level PLA realizes a generalized Boolean function. Then we obtain the sizes of PLA's for various classes of functions.

Lemma 7: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X. In a three-level PLA, if the D array generates all the maxterms of X_i for every i $(i = 1, 2, \cdots, r)$, then an arbitrary function which has a form $X_1^{S_1} \cdot X_2^{S_2} \cdots X_r^{S_r}$ can be generated in each column of the AND array. The width W of the AND array to realize the given function f is equal to the number of terms in a minimal sum-of-products expression for f, and $W \leq 2^{n - \max_i\{n_i\}}$.

By Lemma 7 we have the following.

Theorem 11: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X and the D array generate all the maxterms of X_i for every i. The size of three-level PLA which is sufficient to realize an arbitrary n-variable function is given by the following formula:

$$C(n) = (2n + W)H + W \tag{12}$$

where

$$W = 2^{n - \max_i\{n_i\}},\ H = \sum_{i=1}^{r} 2^{n_i}, \text{ and } n_i = d(X_i).$$

Definition 8: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X. A vector $\mathbf{n} = (n_1, n_2, \cdots, n_r)$ is said to be *a partition vector of the input variables*, where $n_i = d(X_i)$.

Let $\mathbf{n} = \left(\left\lfloor\frac{n}{2}\right\rfloor, 1, 1, \cdots, 1\right)$, where $\left\lfloor\frac{n}{2}\right\rfloor$ denotes the largest integer not exceeding $\frac{n}{2}$, we have the following.

TABLE IX
SIZES OF THREE-LEVEL PLA'S WHICH ARE SUFFICIENT TO REALIZE VARIOUS CLASSES OF FUNCTIONS OF n-VARIABLES

	Arbitrary function		Arbitrary Symmetric function		Parity function	
	Partition	Size	Partition	Size	Partition	Size
n	n	C(n)	n	C(n)	n	C(n)
6	(3,2,1)	288	(3,2,1)	258	(2,2,2)	196
8	(4,2,2)	784	(3,3,2)	572	(2,2,2,2)	392
10	(5,2,2,1)	2216	(4,3,3)	1168	(3,3,2,2)	680
12	(6,2,2,2)	6752	(4,4,4)	2377	(3,3,3,3)	1032
14	(7,2,2,2,1)	22280	(5,5,4)	4670	(3,3,3,3,2)	1600
16	(8,2,2,2,2)	78592	(6,5,5)	8740	(4,3,3,3,3)	2320

Theorem 12: The size of three-level PLA which is sufficient to realize an arbitrary function of n-variable is given by the following formulas:

$$2^n + (3n + 1)\sqrt{2^n} + 2n^2, \qquad \text{when } n \text{ is even.}$$
$$2^n + (2n + 2)\sqrt{2^{n+1}} + 2n(n + 1), \qquad \text{when } n \text{ is odd.}$$

By Theorem 3 we have the following.

Theorem 13: Let $(X_1, X_2, \cdots, X_r)$ be a partition of X, and $f(X)$ be partially symmetric with respect to X_i for all i $(i = 1, 2, \cdots, r)$. The size of three-level PLA which is sufficient to realize $f(X)$ is given by

$$C(n) = (2n + W)H + W$$

where

$$W = \prod_{i=1}^{r} (n_i + 1) \Big/ \left(\max_i \{n_i + 1\}\right),$$
$$H = \sum_{i=1}^{r} 2^{n_i}, \text{ and } \mathbf{n} = (n_1, n_2, \cdots, n_r)$$

is a partition vector.

By Theorem 7 we have the following.

Theorem 14: The size of three-level PLA which is sufficient to realize an n-variable parity function is given by

$$C(n) = (2n + W)H + W \tag{14}$$

where

$$W = 2^{r-1} \text{ and } H = \sum_{i=1}^{r} 2^{n_i}.$$

Table IX shows the value of $C(n)$ calculated from the formulas (12)–(14) and corresponding partition vector $\mathbf{n}$, where $\mathbf{n}$ is chosen to minimize the value of $C(n)$.

VI. CONCLUSIONS

1) Generalized Boolean functions can be directly realized by PLA's with decoders or three-level PLA's.

2) In a PLA with two-bit decoders, the following size is necessary and sufficient to realize an n-variable function when the assignment of the input variables to the decoders is fixed:

for an arbitrary function, $\frac{1}{2}(n + \frac{1}{2})2^n$; for an arbitrary symmetric function, $\frac{2}{3}(n + \frac{1}{2})\sqrt{3}^n$; for a parity function, $(n + \frac{1}{2})\sqrt{2}^n$.

3) A PLA with two-bit decoders requires a smaller array than a two-level PLA. In the case of $n = 10$ and $d = 50$ percent the former is, on the average, 24 percent smaller than the latter.

4) The size of the arrays of PLA's with two-bit decoders can be reduced by optimizing the assignment of the input variables. In the case of $n = 8$ and $d = 40$ percent optimally assigned PLA's are 10 percent smaller than nonoptimally assigned ones.

5) For a three-level PLA, the following size is sufficient to realize an n-variable function ($n = 2r$): $2^n + (3n + 1)\sqrt{2}^n + 2n^2$.

Acknowledgment

The author would like to thank Prof. H. Ozaki, Prof. H. Terada, Prof. K. Kinoshita, and Dr. H. Fujiwara for their encouragement. The author is grateful to Mr. Ishikawa for programming. He would also like to thank the referees for their careful reading of the manuscript.

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Tsutomu Sasao (S'72–M'77) was born in Osaka, Japan, on January 26, 1950. He received the B.E., M.E., and Ph.D. degrees in electronic engineering from Osaka University, Osaka, Japan, in 1972, 1974, and 1977, respectively.

Since 1977 he has been with Osaka University. His research interests are switching theory and logic design. He specializes in the design of PLA and multiple-valued logic circuits.

Dr. Sasao is a member of the Institute of Electronics and Communication Engineers of Japan. He received the NIWA Memorial Award in 1979.