Correspondence

Conservative Logic Elements and Their Universality

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Abstract—**A conservative logic element (CLE) is a multiple-output logic element whose weight of an input vector is equal to that of the corresponding output vector, and fan-out of each output terminal is restricted to one. A CLE is a generalized model of magnetic bubble logic elements, etc. In order to realize an arbitrary function, it is necessary to use constant-supplying elements (CSE's). In this correspondence, we consider the universality of CLE's in relation to the number of CSE's.**

Index Terms—**Logic circuits, logic elements, magnetic bubble logic elements, universality of logic elements.**

I. Introduction

A conservative logic elements (*CLE*) is an n-input m-output logic element whose weight of an input vector is equal to that of the corresponding output vector [1]–[5]. CLE's are generalized models of magnetic bubble logic elements, fluid logic elements, transfer relays, current-mode logic elements without power sources, and so on. It is assumed that fan-out of each output terminal of a CLE is one. In order to realize an arbitrary function, C_1's and C_0's are used as circuit elements besides CLE's. A C_1 always supplies a constant "1" and a C_0 always supplies a constant "0." Both a C_1 and a C_0 are called *constant-supplying elements* (CSE's) (Fig. 1). It is also assumed that fan-out of each CSE is one.

For example, consider the three-input three-output element T shown in Fig. 2. Clearly, this element is a CLE. Suppose that it is desired to realize the function $h(x_1, x_2, x_3) = \overline{x_1 x_2 x_3}$ by using T elements. To realize h, at least one C_1 is necessary because every output function of T satisfies the condition $f_i(0, 0, 0) = 0$ while $h(0, 0, 0) = 1$. Similarly, to realize h, at least one C_0 is necessary because every output function of T satisfies the condition $f_i(1, 1, 1) = 1$ while $h(1, 1, 1) = 0$. On the other hand, it is clear that one C_1 and two C_0's are sufficient to realize the function as shown in Fig. 3. Then, how many CSE's are necessary and sufficient to realize h by using T elements? More generally, how many CSE's are necessary and sufficient to realize an arbitrary function by using CLE's? The purpose of this correspondence is to obtain the necessary and sufficient conditions of CLE's for the realizability of an arbitrary function in relation to the number of CSE's.

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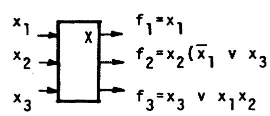
Fig. 1. Constant-supplying elements.



x_1	x_2	x_3	f_1	f_2	f_3
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	0	1
1	1	1	1	1	1

Fig. 2. T element.

II. Definitions and Basic Properties of Conservative Logic Elements

In this section, some definitions and basic properties of CLE's are described. These results will be used in Sections III and IV.

The weight of a vector $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$ is defined as $\|\boldsymbol{a}\| \sum_{i=1}^{n} a_i$.

Definition 2.1: An n-input m-output logic element is said to be a *conservative logic element* (CLE) if it satisfies the following conditions.

1) For any input vector $\boldsymbol{a} \in B^n$, $\|\boldsymbol{a}\| = \|F(\boldsymbol{a})\|$ holds where $B = \{0, 1\}$ and $F(\boldsymbol{a}) = (f_1(\boldsymbol{a}), f_2(\boldsymbol{a}), \cdots, f_m(\boldsymbol{a}))$ is the output vector.
2) Delays of each output can be neglected.

Definition 2.2: A *circuit* satisfies the following conditions (see Fig. 3).

1) It consists of a finite number of components as follows.
   a) Circuit elements (CLE, C_1, and C_0).
   b) External input terminals (as many as the number of input variables).
   c) External output terminals.
2) Any *receiver*[1] is connected to one *sender*,[1] and any sender is connected to one receiver, i.e., the fan-out of each output terminal is one.
3) It has no feedback loops.

Particularly if the circuit has only CLE's as its circuit elements, then the circuit is said to be *a conservative logic circuit* (CLC).

It is clear that the following lemma holds.

[1] Either an input terminal of a CLE or an external output terminal of the circuit is said to be a *receiver*. Either an output terminal of a circuit element or an external input terminal of the circuit is said to be a *sender*.

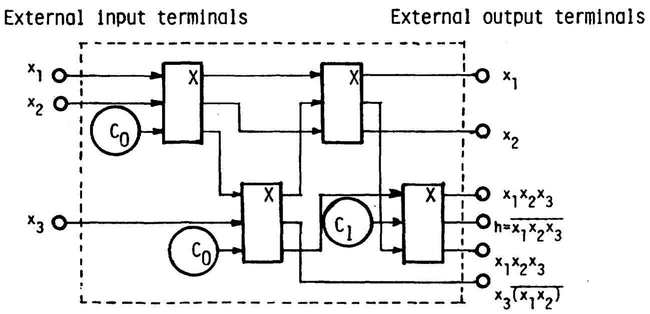


Fig. 3. A circuit which realizes $h = x_1 x_2 x_3$.

Lemma 2.1: In a CLC, for any external input vector, the weight of an external input vector is equal to that of the corresponding external output vector.

So if the inputs of a CLC are all "0," then the outputs are all "0."

Corollary 2.1: The function f such that $f(0, 0, \cdots, 0) = 1$ cannot be realized without C_1.

Definition 2.3: If an n-input m-output CLE ($n \geq 2$) satisfies the following conditions, then the CLE is called an *n-m element.*

1) Every input variable is contained as a proper variable in at least two output functions of the CLE.
2) No output function of the CLE is a constant "0."

By this definition, it follows that an n-m element is a CLE without redundant input/output.

Lemma 2.2: For an n-m element, $2 \leq n \leq m \leq n \cdot 2^{n-1}$.

Lemma 2.3: Only four kinds of 2-m elements exist as shown in Fig. 4. The four elements shown in Fig. 4 are called I_A, I_B, II_A, and II_B elements.[2]

A function f is said to be *linear* if f can be written in a form

$$f(X) = a_0 \oplus a_1 x_1 \oplus a_2 x_2 \oplus \cdots \oplus a_n x_n.$$

A function f is *nonlinear* if f is not linear.

Lemma 2.4: There are at least two nonlinear functions in the outputs of an n-m element.

A function f is said to be *monotone increasing* if $f(\boldsymbol{a}) \geq f(\boldsymbol{b})$ for any $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} \geq \boldsymbol{b}$. A function f is *nonmonotone increasing* if f is not monotone increasing.

Lemma 2.5: There is at least one nonmonotone increasing function in the outputs of an n-m element if $n < m$.

Lemma 2.6: If there is a nonmonotone increasing function in the outputs of an n-m element, then both a variable and its complement can be realized by using $(n-1)$ CSE's.

Lemma 2.7: If there is a nonmonotone increasing function in the outputs of an n-m element, then the function $f(x_1, x_2) = x_1 \bar{x}_2$ can be realized by using at most $(3n-4)$ CSE's.

Lemma 2.8: Let the output lines of a circuit be $l_1, l_2, \cdots, l_m$, which generate the output functions $f_1, f_2, \cdots, f_m$, respectively. By connecting I_A elements to these lines, N_1 "1"'s and N_0 "0"'s can be generated where

$$N_1 = \min_{\boldsymbol{a} \in B^n} \|F(\boldsymbol{a})\|, \qquad N_0 = m - \max_{\boldsymbol{a} \in B^n} \|F(\boldsymbol{a})\|,$$

and

$$F(\boldsymbol{a}) = (f_1(\boldsymbol{a}), f_2(\boldsymbol{a}), \cdots, f_m(\boldsymbol{a})).$$

[2] In the case of magnetic bubble logic elements, only one bubble changes its path in the class I element, while both bubbles change their paths in the class II element [5].

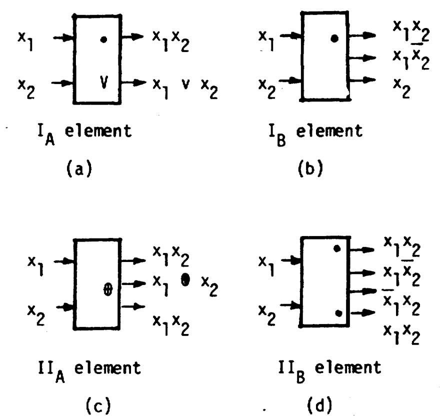


Fig. 4. All the 2-m elements.

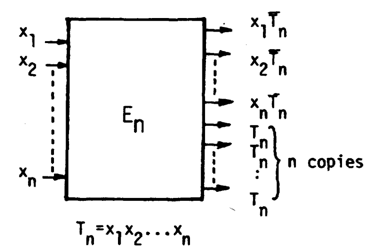


Fig. 5. E_n element.

Lemma 2.9: If $n < m$, then any number of constant "0"'s can be realized by using nC_1's and some n-m elements.

III. Universality of n-m Elements

Definition 3.1: Let $\{\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_r\}$ and $\{\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_s\}$ be two sets of elements and $\{k_1, k_2, \cdots, k_s\}$ be a set of integers. A set $\{\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_r\}$ is said to be universal with $k_1\mathbf{B}_1$'s, $k_2\mathbf{B}_2$'s, $\cdots$, $k_s\mathbf{B}_s$'s if an arbitrary function can be realized with $k_1\mathbf{B}_1$'s, $k_2\mathbf{B}_2$'s, $\cdots$, and $k_s\mathbf{B}_s$'s and an arbitrary number of $\mathbf{A}_j$'s ($j = 1, 2, \cdots, r$).

An n-$2n$ element shown in Fig. 5 is called an $\mathbf{E}_n$ element.

Lemma 3.1: $\{\mathbf{E}_n\}$ is universal with $(2n-2)$ C_1's.

Definition 3.2: Let an $n_1 - m_1$ element $\mathbf{A}_1$ have outputs $f_1^{(1)}, f_2^{(1)}, \cdots, f_{m_1}^{(1)}$, and let an $n_2 - m_2$ element $\mathbf{A}_2$ have outputs $f_1^{(2)}, f_2^{(2)}, \cdots, f_{m_2}^{(2)}$ ($n_1 \geq n_2$, $m_1 \geq m_2$). When we can represent the output functions as

$$f_i^{(1)}(\boldsymbol{a}, \boldsymbol{x}) = f_i^{(2)}(\boldsymbol{x}) \qquad (i = 1, 2, \cdots, m_2)$$

$$f_j^{(1)}(\boldsymbol{a}, \boldsymbol{x}) = \text{constant} \qquad (j = m_2 + 1, \cdots, m_1)$$

by renaming the number of the input variables and output functions of $\mathbf{A}_1$ properly, it is said that $\mathbf{A}_1$ can be used as $\mathbf{A}_2$ where $\boldsymbol{a} = (a_1, a_2, \cdots, a_{n_1 - n_2})$ is a constant vector and $\boldsymbol{x} = (x_1 x_2, \cdots, x_{n_2})$.

Lemma 3.2: If there is a function f_s such that $f_s(e_i) = f_s(e_j) = 1 (i \neq j)$ in the output of an n-m element, then this element can be used either as an I_A element or an II_A element by using $(n-2)C_1$'s.

Corollary 3.1: If an n-m element cannot be used as neither an I_A element nor an II_A element, then by renaming the number of the output functions properly, the output functions can be written as follows:

$$f_i(e_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \qquad (i = 1, 2, \cdots, n).$$

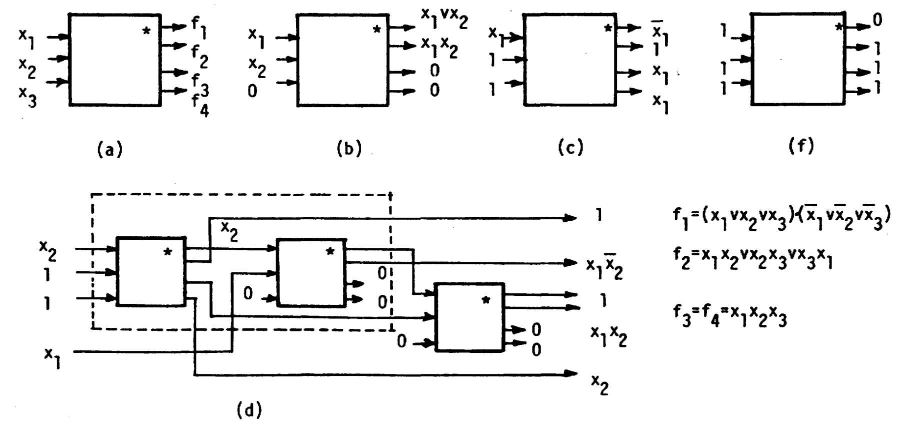


Fig. 6. A circuit which can be used as an I_B element.

So if an element cannot be used as neither an I_A nor an II_A element, then by Corollary 3.1, we can assume the input/output relations without loss of generality as follows. For the input $\mathbf{0} = (0, 0, \cdots, 0)$, the output is $F(\mathbf{0}) = (0, 0, \cdots, 0)$, and for the input $e_i = (0, 0, \cdots, 0, 1_i, 0, \cdots, 0)$, the output is $F(e_i) = (0, 0, \cdots, 0, 1_i, 0, \cdots, 0_n, \cdots, 0_m)$. In other words, for $\|x\| \le 1$, input vector x is equal to the first n-component vector of the corresponding output vector $F(x)$.

Lemma 3.3: For $1 \le \|x\| \le k$, suppose that the input vector x of an n-m element is equal to the first n-component vector of the corresponding output vector $F(x)$. And for an input vector such that $\|a\| = k + 1$, let the first n-component vector of $F(a)$ be b. If $\|a \cdot \bar{b}\| = 1$, then this element can be used as an I_B element, and if $\|a \cdot \bar{b}\| = s \ge 2$, then it can be used as an E_s element.

Example 3.1: Consider the 5–10 element whose output functions are $f_1 = x_1\bar{T}_4$, $f_2 = x_2\bar{T}_4$, $f_3 = x_3\bar{T}_5$, $f_4 = x_4\bar{T}_5$, $f_5 = x_5\bar{T}_5$, $f_6=f_9=f_{10}=\bar{T}_5$, and $f_7=f_8=\bar{T}_4$ where $T_4 = x_1x_2x_3x_4$ and $T_5 = x_1x_2x_3x_4x_5$. Let the input vector be $x = (x_1, x_2, x_3, x_4, x_5)$. When $\|x\| \le 3$, the input vector is equal to the first five-component vector of the corresponding output vector $(f_1(x), f_2(x), f_3(x), f_4(x), f_5(x))$. But for the input vector $a = (1, 1, 1, 1, 0)$, the first five-component vector is $b = (0, 0, 1, 1, 0)$ and $\|a \cdot \bar{b}\| = 2$. So this element can be used as an E_2 element. By assigning $x_3 = x_4 = 1$ and $x_5 = 0$, the output function becomes $f_1 = x_1\bar{T}_2$, $f_2 = x_2\bar{T}_2$, $f_3 = f_4 = 1$, $f_5 = f_6 = f_9 = f_{10} = 0$, and $f_7 = f_8 = \bar{T}_2$ where $T_2 = x_1x_2$.

Lemma 3.4: An n-m element can be used as either an I_B or an E_k element $(2 \le k \le n)$ or it has the function f_s such that $f_s(e_i) = f_s(e_j) = 1 (i \ne j)$ in its outputs.

Lemma 3.5: If an n-m element $\mathbf{A}$ has a function f_s such that $f_s(e_i) = f_s(e_j) = 1 (i \ne j)$ and a nonmonotone increasing function in its outputs, then $\{\mathbf{A}, C_0\}$ is universal with $(3n - 2)C_1$'s.

Theorem 3.1: For an n-m element $\mathbf{A}$, $\{\mathbf{A}\}$ is universal with $(3n - 2)C_1$'s if $n < m$. For an n-m element $\mathbf{B}$, $\{\mathbf{B}, C_0\}$ is universal with $(3n - 2)C_1$'s if and only if $\mathbf{B}$ contains at least one nonmonotone increasing function in its outputs.

Example 3.2: Let us verify the universality of the 3–4 element A shown in Fig. 6(a) where $f = (x_1 \vee x_2 \vee x_3)(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$, $f_2 = x_1x_2 \vee x_2x_3 \vee x_3x_1$, and $f_3 = f_4 = x_1x_2x_3$. As $n < m$, there is a nonmonotone increasing function f_1 in its outputs. We can construct the circuit which produces $f = x_1\bar{x}_2$ according to Lemma 2.7. The complement of a variable can be obtained as Fig. 6(c). The circuit which is inside the broken line of Fig. 6(d) produces the function $f = x_1\bar{x}_2$. Because $f_1(1, 0, 0) = f_1(0, 1, 0) = f_1(1, 1, 0) = 1$, this element can be used as an I_A element as shown in Fig. 6(b). Connecting this I_A element to the circuit shown in Fig. 6(d), we can reproduce a "1." This circuit can be used as an I_B element. A constant "0" can be generated as Fig. 6(f). Hence, $\{\mathbf{A}\}$ is universal with four C_1's. (See [5].)

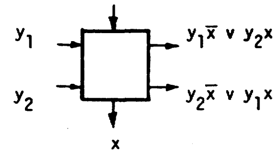


Fig. 7. P element.

IV. Universality of n-n Element

In this section, we consider the universality of an n-n element. The results of this section are useful for the system in which the number of not only C_1's, but also C_0's is important.

Lemma 4.1: For any n-n element $\mathbf{A}$, $\{\mathbf{A}\}$ is not universal without a C_0.

By Lemma 2.3 and Theorem 3.1, a 2–2 element is not universal. First, we will show the universality of the 3–3 element shown in Fig. 7, which is called a P element. To show the universality of the P element, we use the theory of multirail cascade [7].

A *group function* is defined as a mapping from B^m into a finite group. When $m = 1$, it is called an *elementary group function.* A group function $F: B^N \to H$ is said to be decomposable over a group G if it can be expressed as a composition of elementary group functions $\phi_i: B \to G(G \supset H)$. This composition corresponds to a cascade connection of circuits realizing the elementary group function ϕ_i. The overall cascades realize the group function F. It is known that the cyclic group Z_n of odd degree is decomposable [7]. The following lemma is a special case of $n = 3$.

Lemma 4.2: Let $Z_3 = \{I, a, a^2\}$ be a cyclic group of degree 3. $F(X): B^N \to Z_3$ is decomposable as follows:

$$F(X) = g^{x_m} \cdot F_1(X \backslash m) \cdot g^{x_m} \cdot F_2(X \backslash m)$$

where $F_1(X\backslash m)$ and $F_2(X\backslash m)$ denote group functions which do not contain x_m as a variable, $g \in \{S_3 - Z_3\}$, and S_3 is the symmetric group of degree 3.

It should be noted that for any $a \in Z_3$, $g \cdot a^i \cdot g = a^{-i}$.

By using this lemma, we can show that the P element is univer-

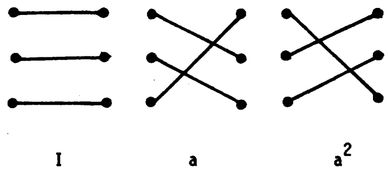


Fig. 8. Circuits which correspond to the elements of Z_3.

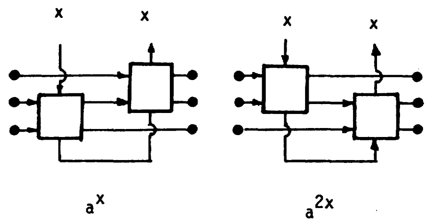


Fig. 9. Circuits which correspond to a^x and a^{2x}.

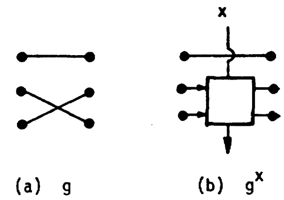


Fig. 10. Circuit which correspond to g and g^x.

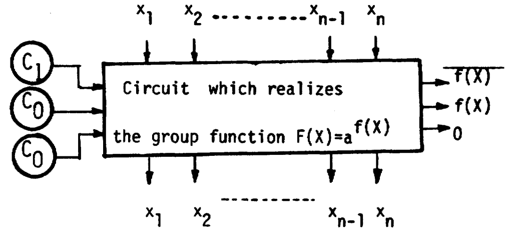


Fig. 11. Realization of logic function $f(X)$.

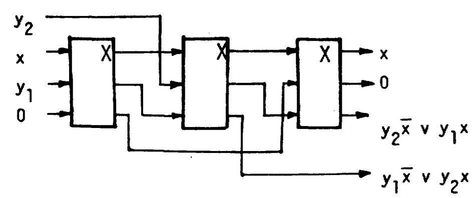


Fig. 12. A circuit which can be used as a P element.

sal. When $x = 0$, the P element propagates two signals y_1 and y_2 straight, while when $x = 1$, it interchanges the signals y_1 and y_2. Let each element of $Z_3 = \{I, a, a^2\}$ correspond to the circuit of Fig. 8. By using two P elements, we can realize a circuit which corresponds to the elementary group function a^x or a^{2x} as shown in Fig. 9. Similarly, if $g \in \{S_3 - Z_3\}$ corresponds to the circuit of Fig. 10(a), then g^x can be realized as shown in Fig. 10(b). Therefore, by Lemma 4.2, any group function $F: B^N \to Z_3$ can be realized by an appropriate cascade connection of the circuits shown in Figs. 8–10. Next, we realize a logic function $f(X): B^N \to B$ by using the circuit which realizes the group function $F(X): B^N \to Z_3$. By connecting CSE's to the circuit which realizes $F(X) = a^{f(X)}$ as shown in Fig. 11, we can obtain a circuit which realizes three logic functions $f(X)$, $\overline{f(X)}$, and "0." In this way, any logic function can be realized by using two C_0's, one C_1, and some P elements.

Lemma 4.3: $\{P\}$ is universal with two C_0's and one C_1.

By using Lemma 4.3 and the results of Sections II and III, we obtain the following results.

Lemma 4.4: If there is at least one nonmonotone increasing function in the outputs of an n-n element **A**, then $\{\mathbf{A}, I_A\}$ is universal with $(s + 2)\, C_0$'s and $(n - s)C_1$'s where s is an integer such that $1 \leq s \leq (n - 2)$.

Lemma 4.5: If an n-n element **A** can be used as an I_A element or an II_A element and contains at least one nonmonotone increasing function in its outputs, then **A** is universal with $(r + 3)C_0$'s and $(2n - r - 2)C_1$'s where r is an integer such that $1 \leq r \leq 2(n - 2)$.

Theorem 4.1: For an n-n element **A**,

1) $\{\mathbf{A}\}$ is universal with n CSE's if **A** can be used as a P element
2) $\{\mathbf{A}\}$ is universal with $(2n + 1)$ CSE's if **A** can be used as an I_A or an II_A element and has a nonmonotone increasing function in its outputs
3) $\{\mathbf{A}\ I_A\}$ is universal with $(n + 2)$ CSE's if and only if **A** has a nonmonotone increasing function in its outputs.

Corollary 4.1: For an n-n element **A**, $\{\mathbf{A}\}$ is universal with $(2n + 1)$ CSE's if **A** has a nonmonotone increasing function and a function such that

$$f_s(e_i) = f_s(e_j) = 1 \qquad (i \neq j).$$

Example 4.1: Let us consider the universality of the **T** element shown in Fig. 2. By setting $x_1 = 1$, the **T** element can be used as an I_A element. **T** has a nonmonotone increasing function f_2 in its outputs. So $\{\mathbf{T}\}$ is universal with seven CSE's by Theorem 4.1. (End of the Example.)

Fig. 12 denotes a circuit which can be used as a P element. Thus, we obtain the following result.

Lemma 4.6: $\{\mathbf{T}\}$ is universal with three C_0's and one C_1.

This result can be applied to the model of magnetic bubble logic studied by Graham [1], and Friedman and Menon [2]. Since the **T** element corresponds to the conditional transfer [2], Lemma 4.6 can be restated by the following.

Corollary 4.2: An arbitrary function can be computed by a program which consists of a sequence of instructions of the form $e = (x_1, x_2, x_3)$ such that $X^e = (X - \{x_2\}) \cup \{x_3\}$ if $x_1, x_2 \in X$ and $x_3 \notin X$, and $X^e = X$ otherwise.

This result is useful for conductor-access magnetic bubble logic elements. An arbitrary logic function can be computed by a program consisting of a sequence of instructions which conserve the number of magnetic bubbles. It should be noted that only four working storages, which correspond to CSE's, are required for the computation in this system.

Acknowledgment

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